

WHITTAKER COEFFICIENTS OF METAPLECTIC EISENSTEIN SERIES

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ABSTRACT. We study Whittaker coefficients for maximal parabolic Eisenstein series on metaplectic covers of split reductive groups. By the theory of Eisenstein series these coefficients have meromorphic continuation and functional equation. However they are not Eulerian and the standard methods to compute them in the reductive case do not apply to covers. For “cominuscule” maximal parabolics, we give an explicit description of the coefficients as Dirichlet series whose arithmetic content is expressed in an exponential sum. The exponential sum is then shown to satisfy a twisted multiplicativity, reducing its determination to prime power contributions. These, in turn, are connected to Lusztig data for canonical bases on the dual group using a result of Kamnitzer. The exponential sum at prime powers is then evaluated for generic Lusztig data. To handle the remaining degenerate cases, the evaluation of the exponential sum appears best expressed in terms of string data for canonical bases, as shown in a detailed example in GL_4 . Thus we demonstrate that the arithmetic part of metaplectic Whittaker coefficients is intimately connected to the relations between these two expressions for canonical bases.

1. INTRODUCTION

The computation of the Whittaker coefficients of Eisenstein series on reductive groups has had far-reaching consequences for the study of automorphic forms. As two prominent examples, Langlands’ computations of the constant terms of Eisenstein series highlighted the role of the dual group in automorphic forms and inspired his functoriality conjectures, while the Casselman-Shalika formula for the coefficients of Borel Eisenstein series at unramified places has been a staple of the Langlands-Shahidi and Rankin-Selberg methods used to obtain analytic properties of L -functions. In this paper, we provide a general method for explicitly computing the Whittaker coefficients of parabolic Eisenstein series on metaplectic covers of reductive groups.

In order to explain the novel features of this more general context, we briefly review the theory for reductive algebraic groups. Let G be a split connected reductive algebraic group defined over a number field F and let P be a standard maximal parabolic subgroup of G with Levi factorization $P = MN$. If π is an automorphic representation of $M(\mathbb{A}_F)$ then one may form the corresponding Eisenstein series on $G(\mathbb{A}_F)$. The constant term of this Eisenstein series is an Euler product and may be expressed in terms of the Langlands L -functions $L(s, \pi, r_i)$ where $\bigoplus_{i=1}^m r_i$ is a decomposition of the adjoint representation of ${}^L G$ on the complexified Lie algebra ${}^L \mathfrak{n}$. The Eisenstein series has continuation in s and satisfies a functional equation and hence so does this constant term. When π is generic the further study

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of the Whittaker coefficients of the Eisenstein series may be used to obtain the continuation of each of these L -functions; this is the Langlands-Shahidi method. In doing so one uses in a critical way the uniqueness of the Whittaker functional to see that the Whittaker coefficients are Eulerian. Then the Casselman-Shalika formula may be applied at almost all places to evaluate the local contributions to the Whittaker coefficients in terms of local Langlands L -functions. By an induction in stages argument, one can similarly study Eisenstein series for non-maximal parabolics.

When G is a split semisimple simply connected algebraic group over F and F contains the n -th roots of unity, Matsumoto [29] defined an n -fold “metaplectic” covering group $\tilde{G}_{\mathbb{A}_F}$ of $G(\mathbb{A}_F)$ that splits over $G(F)$. The group law makes use of the local Hilbert symbol in the completions of F . Following Brylinski and Deligne [13], one may also define metaplectic covers of reductive and non-simply connected groups, sometimes at the expense of requiring that F contain more roots of unity.

In view of the importance of Whittaker coefficients for Eisenstein series on reductive groups, it is natural to ask whether similar explicit formulas exist for metaplectic covers. Already for the n -fold cover of GL_2 one sees that the situation in the metaplectic case is fundamentally different. The constant term of the Eisenstein series in that case is a quotient of Hecke L -functions, but in fact Kubota showed in [25] that each non-degenerate Whittaker coefficient is a Dirichlet series that is an infinite sum of n -th order Gauss sums. When $n > 2$, these Whittaker coefficients are not Eulerian and the space of Whittaker functionals is of dimension greater than 1. However, the metaplectic Eisenstein series still has analytic continuation and functional equation and the Whittaker coefficients do as well. Though these Dirichlet series built out of Gauss sums are not Langlands L -functions, they share all of the same analytic properties while possessing a more complicated arithmetic structure.

This structure was further explored for covers of higher rank groups in the case of Eisenstein series induced from the Borel subgroup by the authors and Daniel Bump [9, 10, 11]. Working in the S -integer formalism (for a finite set of places S , to be described in the next section), we gave a complete answer for the Whittaker coefficients of Borel Eisenstein series on an n -fold cover of SL_{r+1} in [10]. The results are expressed using n -th order Gauss sums whose defining data comes from a surprising source – Kashiwara’s crystal bases.

More precisely, the Whittaker coefficients are indexed by an r -tuple of nonzero S -integers \mathbf{k} which specifies the character of the maximal unipotent at each simple root. To each such \mathbf{k} the associated Whittaker coefficient contains a multiple Dirichlet series roughly of the form

$$\sum_{C_1, \dots, C_r \neq 0} H(C_1, \dots, C_r; \mathbf{k}) |C_1|^{-2s_1} \dots |C_r|^{-2s_r}.$$

Here the s_i are complex variables and the series initially converges when their real parts are sufficiently large. The sum is over S -integers C_i modulo units, and $|C_i|$ denotes the norm of C_i . As noted above, these series are not typically Eulerian but rather are *twisted Euler products* – the coefficients H are not multiplicative, but combine by means of n -th power residue symbols – so that ultimately the entire series is determined by coefficients of the form $H(p^{\ell_1}, \dots, p^{\ell_r}; p^{m_1}, \dots, p^{m_r}) =: H(p^\ell; p^m)$ for a prime p in the S -integers. It is these latter coefficients which may be described using crystal bases. For work on Borel Eisenstein series on covers of other classical groups, see [8, 20, 21].

In this work, we study the Whittaker coefficients of metaplectic Eisenstein series induced from a cover of a maximal parabolic subgroup P . Such series are constructed from metaplectic automorphic representations of lower rank groups. While a number of our methods work in complete generality, our sharpest results are obtained in the case when the parabolic P is “cominuscule” (the omitted simple root appears with multiplicity one in the highest root, see Section 5). In this case, we establish a general formula for these coefficients in terms of exponential sums and the Whittaker coefficients of the inducing data (Theorem 4.1). These coefficients inherit analytic continuation and functional equations from the Eisenstein series, thus giving new examples of Dirichlet series with these analytic properties. The exponential sums appearing in the Whittaker coefficient are then connected to the representation theory of the dual group of G .

The metaplectic cover presents obstacles to the use of standard methods for studying parabolic Eisenstein series on reductive groups. Indeed, local Whittaker models on covers are not unique (and hence the resulting global Whittaker coefficients are not Eulerian). Moreover there is a complicated determination of the metaplectic cocycle which describes the group law on the cover, one that is not amenable to calculations except in rank one.

To explain our resolution of these issues, recall that the Eisenstein series is defined (working over the ring \mathfrak{o}_S of S -integers) as an average over elements $\gamma \in P(\mathfrak{o}_S) \backslash G(\mathfrak{o}_S)$. We show that representatives γ may be parametrized by embeddings of rank one matrices at the roots in the unipotent radical of P . For such matrices, it is possible to systematically handle the cocycle that arises and so this parametrization is well-suited for covers. In it, the order of roots appearing in the product is dictated by a reduced decomposition for w^P , the Weyl group element such that $w_0 = w_M w^P$ where w_M is the long element in the Weyl group of the Levi factor M of P . If P is assumed cominuscule, then representatives γ may be determined using only the bottom rows of the embedded $SL_2(\mathfrak{o}_S)$ matrices, called (c_i, d_i) . Furthermore, we may parametrize double cosets in the flag variety needed in the computation by all non-zero d_i and c_i ’s modulo prescribed products of the d_i .

Though the decomposition depends on a reduced decomposition for w^P , if P is cominuscule such decompositions are unique up to commuting simple reflections (see Remark 5.10). Moreover in Section 8, to any fixed prime p in \mathfrak{o}_S , the valuations of the d_i are shown to match the lengths of edges in the one-skeleton of Kamnitzer’s MV polytopes [23] corresponding to w^P . Thus, by results in [23], the valuations of the d_i are the \mathbf{i}^P -Lusztig data for canonical basis elements in the dual group G^\vee , where \mathbf{i}^P is the reduced word for the decomposition of w^P . Thus we parametrize contributions to the Whittaker coefficient in terms of Lusztig data. Two qualifications are in order. First, Kamnitzer works in the affine Grassmannian, but as the combinatorics of valuations is independent of the base field, we can draw a formal analogy. Second, the results of [23] are presented for the Borel case, but one can do a relative version of [23]. A similar connection was made in the local setting by McNamara [32], who was able to connect a decomposition of the unipotent radical of the standard Borel B directly to MV cycles, initially defined by Mirković and Vilonen [34] in the context of the geometric Satake correspondence.

There is another approach to computing maximal parabolic Eisenstein series on the metaplectic group, using Plücker coordinates on the quotient space $P \backslash G$. Indeed, this was the approach used in prior work of Bump, Hoffstein and the authors [14, 15, 12]. To use such a parametrization on the metaplectic group, one must compute the Kubota symbol, the

root of unity arising from the metaplectic two-cocycle. But computing the Kubota symbol from Plücker coordinates is difficult, and the formulas even in low rank cases are extremely complicated (compare Proskurin [39] and [12], Eqn. (18)). Our approach here avoids this by systematic use of the factorization into rank one subgroups.

There is also another description of Whittaker coefficients of metaplectic Borel Eisenstein series due to Chinta, Gunnells, and Offen [16, 18]. It is constructed as an alternator built from a metaplectic version of the usual Weyl group action, generalizing the Casselman-Shalika formula combined with the Weyl character formula in the case of the trivial cover $n = 1$. This description holds for all covers of quasi-split reductive groups [31]. However, this latter approach does not provide a way to calculate the individual coefficients $H(p^\ell; p^m)$ (for $n = 1$ this is equivalent to passing from the Weyl character formula to the Freudenthal multiplicity formula), and seems restricted to the Borel case.

We expect several applications from the results established here. Indeed, even in the linear algebraic group (i.e. the 1-fold cover) case, this computation is new. As noted above, previous applications of parabolic Eisenstein series for reductive groups avoided evaluating an exponential sum by invoking the fact that automorphic forms are unramified principal series at almost all places and hence evaluated by the Casselman-Shalika formula. Axel Kleinschmidt has informed us that explicit expressions for unipotent integrations of Eisenstein series are required in connection to quantum gravity and the computation of the correction to supergravity via string theory (see Green, Miller, Russo, and Vanhove [22]). While the constant terms give the perturbative term, integrals over unipotent subgroups with non-degenerate characters are expected to arise in the higher order, non-perturbative terms. These integrals should be computable by our methods.

In a different direction, the Dirichlet series we obtain have analytic continuation and functional equation, and these properties may then be used to study the distribution of their coefficients. To give an example, one may induce a pair of automorphic forms on the double cover of GL_2 to GL_4 using the detailed example presented here. A Whittaker coefficient of the resulting Eisenstein series yields a Dirichlet series involving an exponential sum, analyzed below, and a product of Fourier coefficients of the inducing data. These Fourier coefficients in turn are related to central values of quadratically-twisted GL_2 L -functions by work of Waldspurger. Thus our work may be used to study the distribution of pairs of such central values as one varies the quadratic twist.

Lastly, we note that our results are not limited to maximal parabolics. An induction in stages argument similar to that given in [10] allows one to induce from an arbitrary parabolic through a sequence of maximal parabolics. Indeed passing all the way to the Borel subgroup, our results here give generalizations of many of the results in [10] for covers of groups not of type E_8, F_4 , or G_2 . (These three exceptional types are the only cases with no cominuscule roots.) A further analysis of a combinatorial nature is carried out for certain types in [20, 21].

This paper is organized as follows. In Section 2 we set the notation and review information about the metaplectic groups (both local and global). Section 3 explains how to construct a metaplectic Eisenstein series on a split reductive group G by inducing parabolically. This is not as trivial as it sounds, since if the Levi subgroup M of a maximal parabolic subgroup P is of the form $M_1 \times M_2$ where the M_i are linear algebraic groups, then it does not follow that the inverse image of P in the covering group is a direct product. (See for example Takeda [46].) The computation of the Whittaker coefficients of the Eisenstein series is given in Section

4, using a decomposition theorem from Section 5. This order of presentation allows us to complete the discussion of the Whittaker integral before turning to combinatorial questions that occupy the remainder of the paper. Section 5 begins with results for arbitrary reductive groups and their quotients $P \backslash G$. Gradually we place additional assumptions on the parabolic P , arriving at cominuscule parabolics in order to achieve a particularly nice parametrization of double cosets that arise in computing the Eisenstein series. For such parabolics, the Whittaker coefficient may be expressed as a Dirichlet series involving an exponential piece and the Whittaker coefficients of the inducing data.

In Section 6 the exponential sum appearing in the Whittaker function is examined and a twisted multiplicativity theorem is established, reducing these contributions for general argument to the prime power case. This result thus subsumes and generalizes twisted multiplicativity for specific Lie types as in [10, 20]. In Section 7 we briefly illustrate these ideas by working out one case in detail, that of the Eisenstein series on a cover of GL_4 attached to P with Levi factor $GL_2 \times GL_2$. Section 8 concerns canonical bases and the exponential sum. Here we explain the link between the computations above, the Lusztig data, and MV cycles and polytopes. The key point is that Lusztig data appears naturally in the double coset parametrization presented in Section 5, while the value of the exponential sum that appears in the Whittaker coefficient computation is best understood via the polytope built from Kashiwara's string data, as studied by Berenstein-Zelevinsky and Littelmann. It is an important feature that both objects appear naturally at different points in the computation. The problem of giving uniform expressions for the exponential sum at powers of a prime p , valid for all groups and all reduced decompositions for long elements of the Weyl group, is thus closely connected to the appearance of difficult piecewise linear maps in the bijection between Lusztig and string data.

Finally, Sections 9, 10 and 11 concern the evaluation of the exponential sum $H(p^\ell; p^{\mathbf{m}})$ arising in the Whittaker coefficient at powers of a fixed prime p . In Section 9 we prove two results applicable for all cominuscule parabolics. The first (Proposition 9.1) states that the exponential sum has a very regular evaluation for prime powers p^ℓ for ℓ away from certain bounding hyperplanes depending on \mathbf{m} . These hyperplanes conjecturally cut out the associated \mathbf{i} -Lusztig data for the representation of highest weight \mathbf{m} of the dual group (see Remark 9.3). The second result (Proposition 9.2) states that the exponential sum vanishes for ℓ outside the set of Lusztig data bounded in terms of the highest weight $\mathbf{m} + \rho$ of the dual group, where ρ is the Weyl vector of the dual group, with the possible exception of a certain set of hyperplanes. According to [10], this exception does not appear for the nicest (i.e. "Gelfand-Tsetlin") word \mathbf{i} in type A , and one may wonder if it is truly necessary in general.

To address these issues, in Sections 10 and 11 we give a complete evaluation of the prime power contributions to the exponential sum for the GL_4 example initially presented in Section 7. In Section 10, we demonstrate in this example that the exponential sum at prime powers indeed has support at infinitely many lattice points outside the set of Lusztig data for highest weight representations, under the equivalence of the prime power coordinates with Lusztig data. Nevertheless, the total contribution to the Dirichlet series from all basis vectors in a given weight space does vanish for Lusztig data outside the string data polytope (Proposition 10.6). It seems reasonable to expect a similar phenomenon to hold for all cominuscule parabolic, metaplectic Eisenstein series. In Section 11, we demonstrate the

complexity of the evaluation for Lusztig data lying outside the hyperplanes determined by \mathbf{m} , but inside that of $\mathbf{m} + \rho$. For many of these points, we give the evaluation in terms of the corresponding string data (Theorem 11.1). Thus the evaluation of the Whittaker coefficients depends, in an essential way, on the relations between the two most important parametrizations of canonical bases.

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2. METAPLECTIC COVERS

Throughout this paper, let $n \geq 1$ be a fixed integer and let F be a totally complex number field containing a full set of $2n$ -th roots of unity. (The arguments below would work for function fields as well with minor modifications.) For each place v of F , let F_v denote the corresponding completion at v . When v is non-archimedean, let \mathfrak{o}_v denote the ring of integers of F_v and ϖ_v be a local uniformizer. Assume that G is a split reductive group over F . This ensures that, at each place, $G(F_v)$ arises by base extension from a smooth reductive group scheme \mathbf{G} over \mathfrak{o}_v . We first discuss the n -fold metaplectic covers attached to G , both local and global.

2.1. Local fields. The local metaplectic group \tilde{G}_v is a central extension of the group $G(F_v)$ by the group of n -th roots of unity μ_n :

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G}_v \longrightarrow G(F_v) \longrightarrow 1.$$

Central extensions of semisimple, simply connected, algebraic groups over an infinite field k were classified by Brylinski and Deligne [13]. They arise from central extensions of $G(k)$ by $K_2(k)$ as follows. Let T be a maximal split torus of G , and let $Y = \text{Hom}(\mathbb{G}_m, T)$ be the group of cocharacters of T . This comes with a natural action of the Weyl group W of G . Then the central extensions are in bijection with W -invariant symmetric bilinear forms

$$B : Y \times Y \longrightarrow \mathbb{Z}$$

such that $Q(\alpha^\vee) := B(\alpha^\vee, \alpha^\vee)/2 \in \mathbb{Z}$ for all coroots α^\vee . In the case that k is the local field F_v , by composing with the Hilbert symbol map $K_2(F_v) \rightarrow \mu_n$, one realizes the group \tilde{G}_v above.

If v is a complex place, then the Hilbert symbol is identically 1 and the n -fold metaplectic cover \tilde{G}_v is isomorphic to $G(F_v) \times \mu_n$. For the rest of this subsection, suppose instead that v is non-archimedean.

Basic properties of the n -th order local Hilbert symbol may be found in [40]. In particular, we recall that the symbol is a bilinear map $(\cdot, \cdot) : F_v^\times \times F_v^\times \longrightarrow \mu_n$ satisfying

$$(s, t)(t, s) = (t, -t) = (t, 1 - t) = 1 \quad \text{for all } s, t \in F_v^\times.$$

The assumptions on F ensure that $n|(q_v - 1)$, where q_v is the cardinality of the residue field of F_v . This implies that the Hilbert symbol (s, t) , $s, t \in F_v^\times$, is determined from the congruence

$$(s, t) \equiv \left((-1)^{\text{ord}_v(s)\text{ord}_v(t)} s^{\text{ord}_v(t)} t^{-\text{ord}_v(s)} \right)^{(q_v - 1)/n} \pmod{\varpi_v \mathfrak{o}_v},$$

where ord_v is the valuation at v . Moreover, since $2n|(q_v - 1)$, we have the evaluations

$$(-1, t) = 1, \quad (\varpi_v^a, \varpi_v^b) = 1 \quad \text{for any } t \in F_v^\times, a, b \in \mathbb{Z}.$$

In the special case $G = SL_2$, the restriction map $K_2(F_v) \rightarrow \mu_n$ induces a map of abelian groups $\xi : \mathbb{Z} \rightarrow H^2(SL_2(F_v), \mu_n)$. To work over a split reductive group we use a slightly more general result, due to McNamara [33].

Theorem 2.1. *To any split reductive group G over a local field F_v with assumptions as above, and a choice of bilinear form B , there exists a central extension \tilde{G}_v of $G(F_v)$ by μ_n such that, for each root α , the pullback of the central extension under the morphism of group schemes*

$$\iota_\alpha : SL_2 \rightarrow G$$

to a central extension of SL_2 is realized by the cohomology class $\xi(Q(\alpha^\vee))$.

We will work with a two-cocycle $\sigma_v : G(F_v) \times G(F_v) \rightarrow \mu_n$ whose cohomology class in $H^2(G(F_v), \mu_n)$ corresponds to \tilde{G}_v . Then we may realize

$$\tilde{G}_v = \{(g, \zeta) \mid g \in G(F_v), \zeta \in \mu_n\}$$

with multiplication given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \sigma_v(g_1, g_2) \zeta_1 \zeta_2).$$

If $G = SL_2$, following Kubota [25], we have the following explicit formula for a cocycle σ_v^α whose class in $H^2(SL_2(F_v), \mu_n)$ is $\xi(Q(\alpha^\vee))$:

$$\sigma_v^\alpha(g, h) = \left(\frac{x(gh)}{x(g)}, \frac{x(gh)}{x(h)} \right)^{-Q(\alpha^\vee)}, \quad \text{where} \quad x(g) = x \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

For general G , we use the two-cocycle σ_v that matches σ_v^α upon composition with ι_α for each coroot α . Since σ_v is given in terms of local Hilbert symbols, by Hilbert reciprocity we have that if $g, h \in G(F)$ then $\prod_v \sigma_v(g, h) = 1$.

Moreover, given a choice of basis e_1, \dots, e_r for the cocharacter group Y , there is an induced isomorphism $(F_v^\times)^r \simeq T(F_v)$. Then the 2-cocycle σ on a pair of torus elements $s = (s_1, \dots, s_r)$ and $t = (t_1, \dots, t_r)$ may be given explicitly (see p. 304 of [33]) by:

$$\sigma_v(s, t) = \prod_{i \leq j} (s_i, t_j)_{2n}^{q_{i,j}} \quad \text{where} \quad Q\left(\sum_i y_i e_i\right) = \sum_{i \leq j} q_{i,j} y_i y_j. \quad (1)$$

Here we have written $(s, t)_{2n}$ to denote the local Hilbert symbol of order $2n$. Finally, we record two splitting properties of the central extension \tilde{G}_v .

Proposition 2.2. *The extension \tilde{G}_v splits canonically over any unipotent subgroup of $G(F_v)$.*

The splitting is via the trivial section. See, for example, Mœglin-Waldspurger [35, Appendix 1]. Building on work of Brylinski-Deligne (see Section 10.7 of [13]) and Moore ([36], Lemma 11.3), one also has the following splitting (see [33, Theorem 2] and the references cited there).

Proposition 2.3. *Suppose that n is relatively prime to the residue characteristic of F_v . Then \tilde{G}_v splits over the maximal compact subgroup $K_v = G(\mathfrak{o}_v)$.*

In particular, if n is relatively prime to the residue characteristic, following Kubota [25] we may choose a lifting $(k, \kappa_v(k))$ of $G(\mathfrak{o}_v)$ to \tilde{G}_v where, on matrices in $SL_2(\mathfrak{o}_v)$,

$$\kappa_v \left(\iota_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} (c, d)^{-Q(\alpha^\vee)} & \text{if } 0 < |c|_v < 1 \\ 1 & \text{otherwise.} \end{cases} \quad (2)$$

For these F_v the local “Kubota symbol” κ_v then satisfies

$$\sigma_v(g_v, g'_v) = \frac{\kappa_v(g_v)\kappa_v(g'_v)}{\kappa_v(g_v g'_v)} \quad \text{for all } g_v, g'_v \in G(\mathfrak{o}_v). \quad (3)$$

For the remaining non-archimedean places, the cover splits over a subgroup of K_v of finite index.

2.2. S-integers. The metaplectic cover may also be defined over a ring of S -integers \mathfrak{o}_S , the elements of F integral outside a finite set of places S . We require that S contains the archimedean places S_∞ and all places ramified over \mathbb{Q} (in particular including those dividing n), and is large enough such that the ring \mathfrak{o}_S is a principal ideal domain, and so that G is definable over \mathfrak{o}_S as a smooth reductive group scheme, as in Lemma 4.9 of Springer [43]. Let $F_S = \prod_{v \in S} F_v$. The metaplectic extension of $G(F_S)$ by μ_n , denoted \tilde{G}_{F_S} , is the fiber product over μ_n of the local extensions \tilde{G}_v for each $v \in S$ (that is, the quotient of the direct product $\prod_{v \in S} \tilde{G}_v$ by the equivalence relation identifying the central μ_n in each factor). We will use σ to denote the two-cocycle $\sigma = \prod_{v \in S} \sigma_v$ and \mathbf{s} for the section $\mathbf{s}(g) = (g, 1)$. For brevity, we will sometimes write \tilde{G} for \tilde{G}_{F_S} .

We recall the Kubota map κ in the context of the S -integers. If $c, d \in \mathfrak{o}_S$ are coprime, let $(\frac{d}{c}) \in \mu_n$ denote the n -th power residue symbol. (Thus if $c = p$ is prime then this quantity is congruent to $d^{(|p|-1)/n}$ modulo p , where $|\cdot|$ denotes the absolute norm; see [38].) Embed \mathfrak{o}_S in F_S by the diagonal embedding; this gives rise to an embedding of $G(\mathfrak{o}_S)$ in $G(F_S)$.

Lemma 2.4. *There exists a map $\kappa : G(\mathfrak{o}_S) \rightarrow \mu_n$ such that*

$$\kappa(\gamma\gamma') = \sigma(\gamma, \gamma')\kappa(\gamma)\kappa(\gamma'). \quad (4)$$

If α is any positive root then

$$\kappa \left(\iota_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} \left(\frac{d}{c}\right)^{Q(\alpha^\vee)} & \text{if } c \neq 0 \\ 1 & \text{if } c = 0, \end{cases} \quad (5)$$

where $\iota_\alpha : SL_2 \rightarrow G$ is the canonical embedding.

Proof. For $\gamma \in G(\mathfrak{o}_S)$, define

$$\kappa(\gamma) = \prod_{v \notin S} \kappa_v(\gamma).$$

The right-hand side is well-defined owing to (3), which ensures that $\kappa_v(\gamma) = 1$ for almost all $v \notin S$. Using (2) and the relation between the power residue and Hilbert symbols (see for example Neukirch [38]; in Proposition V.3.4 note that the Hilbert symbol there is the inverse of ours), one obtains

$$\kappa \left(\iota_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \prod_{\substack{v \notin S \\ \varpi_v | c}} (\varpi_v, d)^{-Q(\alpha^\vee)\text{ord}_v(c)} = \prod_{\substack{v \notin S \\ \varpi_v | c}} \left(\frac{d}{\varpi_v} \right)^{Q(\alpha^\vee)\text{ord}_v(c)} = \left(\frac{d}{c} \right)^{Q(\alpha^\vee)},$$

since c, d are coprime, thus d is a unit in \mathfrak{o}_v whenever $\varpi_v | c$. Then if $\gamma, \gamma' \in G(\mathfrak{o}_S)$, by Hilbert reciprocity

$$\sigma(\gamma, \gamma') = \prod_{v \in S} \sigma_v(\gamma, \gamma') = \prod_{v \notin S} \sigma_v(\gamma, \gamma')^{-1} = \frac{\kappa(\gamma\gamma')}{\kappa(\gamma)\kappa(\gamma')},$$

as desired. \square

Corollary 2.5. *The map $\iota : G(\mathfrak{o}_S) \rightarrow \tilde{G}$ given by $\iota(\gamma) = (\gamma, \kappa(\gamma))$ is a homomorphism.*

3. EISENSTEIN SERIES ON THE METAPLECTIC GROUP

We now define the maximal parabolic Eisenstein series on the n -fold cover \tilde{G} of $G(F_S)$ that will be the focus of the remainder of the paper. Let P be a standard maximal parabolic subgroup of G and let M be the Levi subgroup of P . We shall suppose that this Levi subgroup factors as $M = M_1 \times M_2$ where M_i are linear algebraic groups.

If H is any subgroup of $G(F_S)$, let $\tilde{H} = \{(h, \zeta) \mid h \in H, \zeta \in \mu_n\}$ denote the full inverse image of H in \tilde{G} . In particular, we have the subgroups \tilde{M}_i , $i = 1, 2$ and \tilde{M} of \tilde{G} . For a subgroup H of $G(F_S)$, let

$$H^0 = \{h \in H \mid \det(h) \in \Omega\} \quad \Omega = \mathfrak{o}_S^\times (F_S^\times)^n.$$

The subgroup Ω is maximal isotropic with respect to the Hilbert symbol. The maximality may be deduced from Proposition 8 of Section XIII.5 of [47]. We also introduce the subgroup $M_0 := M_1^0 \times M_2^0$ of M^0 . For any two subgroups H_1, H_2 of $G(F_S)$, let $\tilde{H}_1 \times_{\mu_n} \tilde{H}_2$ denote the fiber product over μ_n . Then in general $\tilde{M}_1 \times_{\mu_n} \tilde{M}_2$ does *not* embed in \tilde{G} , owing to the nature of the metaplectic cocycle. (For example in the case $G = GL_r$, this may be seen using block compatibility as in Banks, Levy and Sepanski [2].) However, its subgroup $\tilde{M}_1^0 \times_{\mu_n} \tilde{M}_2^0$ does embed in \tilde{G} , and is canonically isomorphic to the subgroup \tilde{M}_0 , which is of finite index in \tilde{M} . It is clear the cover splits over $\tilde{T}_1^0 \times_{\mu_n} \tilde{T}_2^0$ using (1). To extend the result to covers of M_i , by the Bruhat decomposition it suffices to understand the cocycle coming from a torus element of T_1 and a Weyl group element from M_2 . This is trivial according to [29] (see also p. 21 of [30]).

Recall that S_∞ denotes the set of archimedean places. Let S_{fin} denote the set of finite places in S , and let $F_\infty = \prod_{v \in S_\infty} F_v$, $F_{S_{\text{fin}}} = \prod_{v \in S_{\text{fin}}} F_v$. Thus we may write $H^0 = H_{\text{fin}}^0 \times H(F_\infty)$ where $H(F_\infty) = \prod_{v \in S_\infty} H(F_v)$. Since the cocycle is trivial at any complex place, we also have a natural factorization $\tilde{H}^0 = \tilde{H}_{\text{fin}}^0 \times H(F_\infty)$. For the same reason, we also have a natural factorization $\tilde{G} = \tilde{G}_{\text{fin}} \times G(F_\infty)$.

3.1. Inducing in Stages. We fix an embedding of μ_n into \mathbb{C}^\times throughout this paper. Let (π_i, V_i) for $i = 1, 2$ be genuine automorphic representations of \tilde{M}_i which are unramified outside S . This requires a bit of care. Indeed, if σ is a two-cocycle representing a given cohomology class in $H^2(G(F_S), \mu_n)$, let σ_i for $i = 1, 2$ be the restriction of σ to $M_i(F_S)$. Then the class of σ_i is in $H^2(M_i(F_S), \mu_n)$. However, it is possible that the class of this cocycle lies in a smaller group $H^2(M_i(F_S), \mu_{n'})$ where n' is a proper divisor of n . (For example, if P is a parabolic subgroup of $G = GSp_{2a+b}$ such that $M \cong GL_a \times GSp_b$, the cocycle σ restricted to GL_a lies in $H^2(GL_a(F_S), \mu_{n/(n,2)})$.) In this case, a genuine automorphic representation

on \widetilde{M}_i corresponds to one on the n' -fold cover, extended to the n -fold cover obtained by the inclusion $H^2(M_i(F_S), \mu_{n'}) \subseteq H^2(M_i(F_S), \mu_n)$.

Let (π_i^0, V_i) denote the restriction of π_i to \widetilde{M}_i^0 . Then $\pi_1^0 \otimes \pi_2^0$ gives a representation of $\widetilde{M}_0 \simeq \widetilde{M}_1^0 \times_{\mu_n} \widetilde{M}_2^0$. (Indeed, since both π_i^0 are genuine, the product $\pi_1^0 \otimes \pi_2^0$ is well-defined modulo \sim_{μ_n} .) We will define a representation of \widetilde{G} by inducing in stages, first from \widetilde{M}_0 to \widetilde{M} and then parabolically inducing from \widetilde{M} to \widetilde{G} .

For each $v \in S_{\text{fin}}$, let K_v be a compact open subgroup of $G(F_v)$ such that \widetilde{G}_v splits over K_v (as noted in Proposition 2.3, we may take $K_v = G(\mathfrak{o}_v)$ when n is relatively prime to the residual characteristic of F_v). For each $v \in S_\infty$ let K_v be a maximal compact subgroup. Let $K = \prod_{v \in S} K_v$ and $K_\infty = \prod_{v \in S_\infty} K_v$. We regard K as contained in \widetilde{G} via the product of the local splittings.

For the first step, the induced representation $V_{\widetilde{M}}$ of \widetilde{M} is constructed as follows:

$$V_{\widetilde{M}} := \text{Ind}_{\widetilde{M}_0}^{\widetilde{M}}(\pi_1^0 \otimes \pi_2^0) = \{\phi : \widetilde{M} \longrightarrow V_1 \otimes V_2 \mid \phi \text{ genuine and } K \cap \widetilde{M}\text{-finite, and} \\ \phi(\tilde{h}\tilde{m}) = (\pi_1^0 \otimes \pi_2^0)(\tilde{h}) \cdot \phi(\tilde{m}) \forall \tilde{h} \in \widetilde{M}_0, \tilde{m} \in \widetilde{M}\}.$$

As usual \widetilde{M} acts on $V_{\widetilde{M}}$ by the right regular representation; we denote this $\pi_{\widetilde{M}}$ below. Note that any $\phi \in V_{\widetilde{M}}$ is determined by its restriction to $\widetilde{M}_{\text{fin}}$ since $M(F_\infty)$ embeds in \widetilde{M}_0 .

Let us explain how to construct functions in $V_{\widetilde{M}}$ concretely. Let R be a set of coset representatives for $M_0 \backslash M(F_S)$. For later use, we require this set of representatives to be in $T(F_S)$ and the equal to the identity element of $T(F_v)$ for $v \in S_\infty$. Let

$$\mathcal{M}_{P,R}(\Omega) := \left\{ \Psi : M \rightarrow \mathbb{C} \mid \Psi(hr) = \sigma(h, r)^{-1} \Psi(r) \text{ for all } h \in M_0, r \in R \right\}. \quad (6)$$

Note that

$$\sigma(h, m) = \sigma_1(h_1, m_1) \sigma_2(h_2, m_2)$$

if $h = (h_1, h_2)$ and $m = (m_1, m_2)$ are in M_0 by the block compatibility property of σ and the isotropy property of Ω . In the special case $M = GL_1 \times GL_1$ this factor is identically 1, so the space is independent of the choice of R (compare [7]). Let $f_i \in V_i$ for $i = 1, 2$, and $\Psi \in \mathcal{M}_{P,R}(\Omega)$. Define the $V_1 \otimes V_2$ -valued function

$$\phi_{f_1, f_2, \Psi}(\tilde{m}) := \zeta \Psi(m) \pi_1^0(\mathbf{s}(m_1)) \otimes \pi_2^0(\mathbf{s}(m_2)) \cdot (f_1 \otimes f_2) \quad (7)$$

for $\tilde{m} = (m, \zeta) = ((m_1, m_2)r, \zeta)$ with $m_i \in M_i^0$, $r \in R$, $\zeta \in \mu_n$. (We frequently write $\phi(\tilde{m})$ and omit the subscripts.)

Proposition 3.1. *The space $V_{\widetilde{M}}$ is spanned by the functions $\phi_{f_1, f_2, \Psi}$ with $f_i \in V_i$, $i = 1, 2$ and $\Psi \in \mathcal{M}_{P,R}(\Omega)$. Moreover, this spanning set does not depend on the choice of coset representatives R (up to multiplication by nonzero constants).*

Proof. If $\tilde{h} = (h, \eta) = ((h_1, h_2), \eta) \in \widetilde{M}_1^0 \times_{\mu_n} \widetilde{M}_2^0$ and $\tilde{m} = (m, \zeta) = ((m_1, m_2)r, \zeta)$ with $m_i \in M_i^0$, $r \in R$, $\zeta \in \mu_n$, then we have $\tilde{h}\tilde{m} = ((h_1 m_1, h_2 m_2)r, \zeta \eta \sigma(h, m))$. Also using the cocycle property it is not hard to check that $\Psi \in \mathcal{M}_{P,R}$ if and only if

$$\Psi(hm) = \sigma(h, m)^{-1} \sigma(h, mr^{-1}) \Psi(m) \text{ for all } h \in M_0, m \in M \text{ with } m \in M_0 r, r \in R.$$

Thus we see that if $\phi = \phi_{f_1, f_2, \Psi}$, then

$$\begin{aligned} \phi(\tilde{h}\tilde{m}) &= \zeta \eta \sigma(h, m) \Psi(hm) \pi_1^0(\mathbf{s}(h_1 m_1)) \otimes \pi_2^0(\mathbf{s}(h_2 m_2)) \cdot (f_1 \otimes f_2) = \\ &= \eta \pi_1^0(\mathbf{s}(h_1)) \otimes \pi_2^0(\mathbf{s}(h_2)) \cdot \phi(\tilde{m}). \end{aligned}$$

Hence $\phi_{f_1, f_2, \Omega} \in V_{\tilde{M}}$.

Conversely, the space $V_{\tilde{M}}$ is spanned by the functions $\phi \in V_{\tilde{M}}$ such that $\phi(\mathbf{s}(r))$ is a pure tensor for each $r \in R$; write $\phi(\mathbf{s}(r)) = f_{1,r} \otimes f_{2,r}$ with $f_{i,r} \in V_i$. Then $\phi(\tilde{h}\mathbf{s}(r)) = (\pi_1^0 \otimes \pi_2^0)(\tilde{h}) \cdot f_{1,r} \otimes f_{2,r}$ for each $\tilde{h} \in \tilde{M}_0$. If $\tilde{m} = ((m_1, m_2)r, \zeta)$ with $m_i \in M_i^0$, $r \in R$, $\zeta \in \mu_n$, then with $\tilde{h} = ((m_1, m_2), \sigma((m_1, m_2), r)^{-1}\zeta)$ we see that $\tilde{m} = \tilde{h}\mathbf{s}(r)$. Thus

$$\phi(\tilde{m}) = \sigma((m_1, m_2), r)^{-1} \zeta \pi_1^0(\mathbf{s}(m_1)) \otimes \pi_2^0(\mathbf{s}(m_2)) \cdot f_{1,r} \otimes f_{2,r}.$$

For each $r \in R$, define $\Psi_r(m) = \sigma(m_0, r)^{-1}$ when $m = m_0 r$ with $m_0 \in M_0$, and $\Psi_r(m) = 0$ if $m \notin M_0 r$. Then it is immediate that $\Psi_r \in \mathcal{M}_{P,R}(\Omega)$. We see that

$$\phi = \sum_{r \in R} \phi_{f_{1,r}, f_{2,r}, \Psi_r}.$$

Thus we conclude that the space $V_{\tilde{M}}$ is indeed spanned by functions of the form (7).

Last, let us show that the set of functions $\phi_{f_1, f_2, \Psi}$ with $f_1 \in V_1$, $f_2 \in V_2$ and $\Psi \in \mathcal{M}_{P,R}(\Omega)$ does not depend on the choice of coset representatives R . To see this, let $r \in R$, and let R' be a set of coset representatives in which the representative r is replaced by hr with $h \in M_0$. We add superscripts to indicate the choice of coset representatives. If $h = (h_1, h_2)$ and if $m = m_0 r \in M_0 r$ then $\Psi_{hr}^{R'}(m) = \sigma(m_0 h^{-1}, hr)^{-1}$. Using the cocycle property, it is then easy to check that

$$\phi_{f_1, f_2, \Psi_{hr}^{R'}}^{R'} = \sigma(h^{-1}, hr)^{-1} \phi_{f'_1, f'_2, \Psi_r^R}^R$$

with $f'_1 = \pi_1^0(\mathbf{s}(h_1))^{-1} f_1$, $f'_2 = \pi_2^0(\mathbf{s}(h_2))^{-1} f_2$. \square

Next we parabolically induce up to \tilde{G} . Suppose that P has Levi decomposition $P = MU^P$. Then $\tilde{P} = \tilde{M}U^P$, where we have identified U^P with its canonical embedding $\mathbf{s}(U^P)$. Let δ_P be the modular character of M and for $s \in \mathbb{C}$, let χ_s denote the character $\chi_s = \delta_P^s$. Extend these characters trivially to the cover by composing with the projection $\tilde{M} \rightarrow M(F_S)$ (thus they are non-genuine characters of \tilde{M}). Let $\pi_s = \chi_s \delta_P^{1/2} \pi_{\tilde{M}}$, and extend this to a representation of \tilde{P} that is trivial on U^P . The space of the induced representation is

$$\text{Ind}_{\tilde{P}}^{\tilde{G}}(\pi_s) = \{\varphi : \tilde{G} \longrightarrow V_{\tilde{M}} \mid \varphi \text{ smooth and}$$

$$\varphi(\tilde{m}u\tilde{g}) = \pi_s(\tilde{m}) \cdot \varphi(\tilde{g}) \forall \tilde{m} \in \tilde{M}, u \in U^P, \tilde{g} \in \tilde{G}\}.$$

For later use, note that since $\tilde{P} = \tilde{P}_{\text{fin}} \times P(F_\infty)$, the representation π_s factors as $\pi_s = \pi_{s, \text{fin}} \otimes \pi_{s, \infty}$.

3.2. Explicit Test Vectors. Let B be a standard Borel subgroup of G such that $B \subseteq P$ and let $U := U^B$ denote the unipotent radical of B . Let U^P denote the unipotent radical of P and $U_M := U \cap M$, where M is the Levi subgroup of P . Let $w_0 \in G$ denote (a representative for) the long element of the Weyl group W . It may be factored as $w_0 = w_M w^P$ where w_M is the long element of $W_M = W \cap M$, the Weyl group of M , and w^P is a minimal length coset representative for the long element in $W_M \backslash W$.

Recall that the finite set of places $S = S_{\text{fin}} \cup S_{\infty}$. Let $\mathfrak{o}_{\text{fin}} := \prod_{v \in S_{\text{fin}}} \mathfrak{o}_v$. For each $v \in S_{\text{fin}}$, let \mathfrak{a}_v be an ideal of \mathfrak{o}_v which is sufficiently large that if $a \in F_v$, $a - 1 \in \mathfrak{a}_v$, then a is an n -th power in F_v^\times . Let $\mathfrak{a} := \prod_{v \in S_{\text{fin}}} \mathfrak{a}_v$, an ideal of $\mathfrak{o}_{\text{fin}}$. We further require that \mathfrak{a} is divisible by the conductors of π_1 and π_2 in $\mathfrak{o}_{\text{fin}}$. Let $U^P(\mathfrak{a})$ be the subgroup of $U^P(F_{S_{\text{fin}}})$ generated by the one-parameter subgroups at roots in U^P with elements in \mathfrak{a}_v for all $v \in S_{\text{fin}}$. These definitions of groups and elements depend on a realization of the algebraic group G ; this is reviewed briefly in Section 5.

Fix coset representatives R for $M_0 \backslash M(F_S)$ as above. Given automorphic forms $f_i \in V_i$, $i = 1, 2$ and given Ψ in $\mathcal{M}_{P,R}(\Omega)$ (see (6)), recall that $\phi_{f_1, f_2, \Psi} \in V_{\tilde{M}}$ is defined by (7). Using this function, we may construct a function $\varphi_{s, \mathfrak{a}}$ in $\text{Ind}_{\tilde{P}}^{\tilde{G}}(\pi_s)$ associated to sufficiently large ideals \mathfrak{a} of $\mathfrak{o}_{\text{fin}}$ as follows. For simplicity, we assume that the vectors f_i in V_i are fixed under the standard maximal compact subgroups of $M_i(F_{\infty})$. (More generally, one may proceed along the lines of Section 2 of [41], by acting on the right by a suitably chosen finite dimensional representation of the compact subgroup of $M(F_{\infty})$.) To define $\varphi_{s, \mathfrak{a}}$, factor $\tilde{g} = (\tilde{g}_{\text{fin}}, g_{\infty})$, and let

$$\varphi_{s, \mathfrak{a}}(\tilde{g}) = \begin{cases} \pi_s((\tilde{m}_{\text{fin}}, m_{\infty})) \cdot \phi_{f_1, f_2, \Psi} & \text{if } \tilde{g}_{\text{fin}} = \tilde{m}_{\text{fin}} u_{\text{fin}} w^P n_{\text{fin}} \in \tilde{P}_{\text{fin}} w^P U^P(\mathfrak{a}) \\ & \text{and } g_{\infty} = m_{\infty} u_{\infty} k_{\infty} \in G(F_{\infty}), \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Here $\tilde{m}_{\text{fin}} \in \tilde{M}_{\text{fin}}$, $u_{\text{fin}} \in U^P(F_{S_{\text{fin}}})$, and $n_{\text{fin}} \in U^P(\mathfrak{a})$. Moreover, $m_{\infty} \in M(F_{\infty})$, $u_{\infty} \in U(F_{\infty})$ and $k_{\infty} \in K_{\infty}$.

Proposition 3.2. *The function $\varphi_{s, \mathfrak{a}}$ is an element of $\text{Ind}_{\tilde{P}}^{\tilde{G}}(V_{\tilde{M}})$.*

Proof. It follows immediately from the definition that $\varphi_{s, \mathfrak{a}}$ satisfies the correct transformation property under left multiplication. To check smoothness, if $v \in S_{\text{fin}}$ then make use of the bijective “generalized Plücker coordinate” map $\lambda : P(F_v) \backslash G(F_v) \rightarrow \mathbb{P}(V)$ where $\mathbb{P}(V)$ is the projective space attached to a certain finite dimensional F_v -vector space V . Moreover, we may identify V with F_v^t for some t such that $\lambda(w_P) = [1 : 0 : \dots : 0]$ and such that the image of $\tilde{P}_v w^P U^P(\mathfrak{a}_v)$ is precisely the points of the form $[1 : a_1 : \dots : a_{t-1}]$ with $a_{i,v} \in \mathfrak{a}_v$, $1 \leq i \leq t-1$. Let $K(\mathfrak{a}_v)$ be the principal congruence subgroup of K_v modulo \mathfrak{a}_v . If $k \in K(\mathfrak{a}_v)$, then since k is congruent to the identity modulo \mathfrak{a}_v , $\lambda(P(F_v)g)$ is of the form $[1 : a_1 : \dots : a_{t-1}]$ with $a_{i,v} \in \mathfrak{a}_v$ for $1 \leq i \leq t-1$ if and only if $\lambda(P(F_v)gk)$ is of this form. To see that $f_{\mathfrak{a}}$ is locally constant at v , we look at the decompositions of g and of gk for such k . Suppose that $g = muw_0n$ and $gk = m'u'w_0n'$ with $m, m' \in M(F_v)$, $u, u' \in U^P(F_v)$, $n, n' \in U^P(\mathfrak{a}_v)$. Then $k = g^{-1}gk = n^{-1}w_0^{-1}u^{-1}m^{-1}m'u'w_0n'$. Since $n, n' \in K(\mathfrak{a}_v)$ and w_0 normalizes $K(\mathfrak{a}_v)$, it follows that $u^{-1}m^{-1}m'u' \in K(\mathfrak{a}_v)$, whence $m^{-1}m' \in K(\mathfrak{a}_v) \cap M(F_v)$. Moreover, the cocycles in this last computation do not change when we push this from $G(F_v)$ to \tilde{G}_v , since if $k \in \tilde{P}_v \cap K(\mathfrak{a}_v)$ then its block determinants are n -th powers. That is, if $\tilde{g} = \tilde{m}uw_0n$ and $\tilde{g}k = \tilde{m}'u'w_0n'$ then $\tilde{m}' = \tilde{m}k'$ for some $k' \in K(\mathfrak{a}_v) \cap \tilde{M}(F_v)$, and also $K(\mathfrak{a}_v) \cap \tilde{M}_v$ is naturally isomorphic to $(K(\mathfrak{a}_v) \cap \tilde{M}_{1,v}) \times (K(\mathfrak{a}_v) \cap \tilde{M}_{2,v})$. It follows that $\varphi_{s, \mathfrak{a}}(\tilde{g}) = \varphi_{s, \mathfrak{a}}(\tilde{g}k)$ for all $k \in K(\mathfrak{a}_v)$, that is, that $\varphi_{s, \mathfrak{a}}$ is locally constant at v . If instead $v \in S_{\infty}$, then since the local cocycle at an archimedean place is trivial, it follows immediately from the definition that $\varphi_{s, \mathfrak{a}}(\tilde{g}) = \varphi_{s, \mathfrak{a}}(\tilde{g}k)$ for all $k \in K_v$. \square

3.3. Construction of the Eisenstein Series. Suppose that $\varphi_s \in \text{Ind}_{\tilde{P}}^{\tilde{G}}(\pi_s)$ for s in an open set, that φ_s is continuous as a function of s and that φ_s restricted to K is independent of s . (These assumptions are true for the test vectors described above.)

Consider $\varphi_s(\iota(\gamma)\tilde{g})$ where $\gamma \in P(\mathfrak{o}_S)$ and $\tilde{g} \in \tilde{G}$. Write $\gamma = hu$ with $h \in M(\mathfrak{o}_S)$ and $u \in U^P(\mathfrak{o}_S)$. By the defining property of $\text{Ind}_{\tilde{P}}^{\tilde{G}}(\pi_s)$ and the definition of δ_P , we have

$$\varphi_s(\iota(\gamma)\tilde{g}) = \pi_s(\iota(h))\varphi_s(\tilde{g}) = \pi_{\tilde{M}}(\iota(h))\varphi_s(\tilde{g}).$$

Moreover, $\iota(h) \in \widetilde{M}_0$. Suppose $\tilde{m} \in \widetilde{M}$. Then, using the definition of $V_{\tilde{M}}$, we obtain

$$\varphi_s(\iota(\gamma)\tilde{g})(\tilde{m}) = \varphi_s(\tilde{g})(\iota(h)\tilde{m}).$$

Let $\Lambda : V_1 \otimes V_2 \rightarrow \mathbb{C}$ be the functional that evaluates automorphic forms at the identity. If $\phi \in V_{\tilde{M}}$ and $\tilde{m} \in \widetilde{M}$ let us write $\phi_{\tilde{m}}$ in place of $\phi(\tilde{m})$. Then $\phi_{\tilde{m}} \in V_1 \otimes V_2$ and

$$\Lambda(\phi_{\tilde{h}\tilde{m}}) = \phi_{\tilde{m}}(\tilde{h}) \quad \text{for all } \tilde{h} \in \widetilde{M}_0, \tilde{m} \in \widetilde{M}.$$

Denote the application of Λ to φ_s by

$$\varphi_s(\iota(\gamma)\tilde{g})(\tilde{m})(e) = \varphi_s(\tilde{g})(\tilde{m})(\iota(h)). \quad (9)$$

From now on we assume that $\varphi_s(\tilde{g})(\tilde{m})$ is $\iota(M(\mathfrak{o}_S))$ -fixed. For given $f_i \in V_i$, this can always be arranged for the test vectors defined above by choosing S sufficiently large. Writing ξ_s for the composition of φ_s with Λ , we obtain

$$\xi_s(\iota(\gamma)\tilde{g}) = \xi_s(\tilde{g}) \quad \text{for all } \gamma \in P(\mathfrak{o}_S), \tilde{g} \in \tilde{G}$$

(an equality of complex-valued functions on \widetilde{M}). Let us observe that ξ_s restricts to an automorphic form of \widetilde{M} . Indeed, (9) shows that ξ_s is a function on \widetilde{M} that is left-invariant under $M(\mathfrak{o}_S)$. Applying a second functional Λ' given by evaluation at the identity of \widetilde{M} , then $f_s := \Lambda' \circ \xi_s$ is a complex-valued function on \tilde{G} . Hence we may construct an Eisenstein series from f_s .

Given a function f_s as above, we define the Eisenstein series

$$E_{f,s}(g) = \sum_{\gamma \in P(\mathfrak{o}_S) \backslash G(\mathfrak{o}_S)} f_s(\iota(\gamma)g). \quad (10)$$

It converges for $\Re(s)$ sufficiently large. We sometimes suppress the dependence on f and write the series simply as $E_s(g)$.

Proposition 3.3 (Mœglin-Waldspurger). *$E_{f,s}$ possesses analytic continuation in s to a meromorphic function on \mathbb{C} with functional equation as $s \mapsto 1 - s$.*

Proof. We show how to use the strong approximation theorem to prove that a test vector given as a function on \tilde{G} can be written as an adelic test vector satisfying the conditions imposed by Mœglin-Waldspurger [35].

Using the strong approximation theorem, it follows that the adelic metaplectic cover $\tilde{G}_{\mathbb{A}}$ of $G(\mathbb{A})$ has the decomposition

$$\tilde{G}_{\mathbb{A}} = G(F)\tilde{G}_{F_S}K^S$$

where K^S denotes the product of $G(\mathfrak{o}_v)$ at $v \notin S$. Here $G(F)$ and K^S have been identified with their embeddings as subgroups of the metaplectic group $\tilde{G}_{\mathbb{A}}$, as the cocycle is trivial on $G(F)$ by Hilbert reciprocity and splits over K^S according to the Kubota map.

Suppose first that f_s is K^S -invariant. Then we may lift the functions f_s on \widetilde{G}_{F_S} appearing in $\pi(s)$ to functions $f_s^{\mathbb{A}}$ on $\widetilde{G}_{\mathbb{A}}$, as follows. For any $\tilde{g}_{\mathbb{A}} \in \widetilde{G}_{\mathbb{A}}$, we write

$$\tilde{g}_{\mathbb{A}} = \gamma \tilde{g} k, \quad \gamma \in G(F), \tilde{g} \in \widetilde{G}_{F_S}, k \in K^S$$

and then define $f_s^{\mathbb{A}}$ via the formula

$$f_s^{\mathbb{A}}(\tilde{g}_{\mathbb{A}}) = f_s(\tilde{g}).$$

This function can be used to write the Eisenstein series $E_{f,s}$ adelically, and the results of [35] then apply. (A similar formula appears for Borel Eisenstein series on covers of the general linear group in [24], p. 124.)

More generally, since functions in $V_{\widetilde{M}}$ are $K \cap \widetilde{M}$ -finite, we may find a finite dimensional representation τ of K^S and integrate against f_s to extend it to an operator-valued function \hat{f}_s that transforms under the right by τ , as in [42], Lecture 2. This function may be extended to an adelic function by $\hat{f}_s^{\mathbb{A}}(\tilde{g}_{\mathbb{A}}) = \tau(k)\hat{f}_s(\tilde{g})$. The matrix coefficients may then be used to make a test vector with the desired properties, and the properties of the original series follow. See Langlands [26], Chapter 4 and Shahidi [42], Section 2.1, especially Equation (2.7). \square

We note that the section $f_s^{\mathbb{A}}$ is not factorizable. Indeed, even inducing from the Borel subgroup, it is necessary to introduce an infinite sum of factorizable sections. For type A, this is the passage from f_* to f_0 found in [24], p. 109.

4. WHITTAKER COEFFICIENTS OF THE EISENSTEIN SERIES

We turn to the computation of the Whittaker coefficients of the Eisenstein series. Recall that B denotes a Borel subgroup of G such that $B \subseteq P$ and U denotes the unipotent radical of B . For convenience, since $U(F_S)$ embeds canonically in \widetilde{G} , if $u \in U(F_S)$ we write $u \in \widetilde{G}$ instead of $(u, 1) \in \widetilde{G}$. Let ψ be a non-degenerate character of $U(F_S)$ that is trivial on $U(\mathfrak{o}_S)$. Then the ψ -Whittaker coefficient of the Eisenstein series is given by the integral

$$\int_{U(\mathfrak{o}_S) \backslash U(F_S)} E_{f,s}(u\tilde{g}) \overline{\psi(u)} du, \quad \tilde{g} \in \widetilde{G}.$$

In this section we evaluate these coefficients, modulo some decomposition theorems that are treated in detail in later sections. The sharpest of these theorems require that G is cominuscule – see Definition 5.6 – so we make these assumptions henceforth. Since \widetilde{G} acts on the right, without loss we take $\tilde{g} = 1$ from now on. To evaluate, we suppose that $\Re(s)$ is sufficiently large, and we replace the Eisenstein series by its definition as a sum. Since ψ is non-degenerate, it is not hard to check that the Whittaker coefficient is supported on summands arising from the relative “big Bruhat cell” $P(F_S)w_0U^P(F_S) \cap G(\mathfrak{o}_S)$, where $w_0 \in G(\mathfrak{o}_S)$ is a representative for the long Weyl group element. Using the usual unfolding arguments, together with the fact that $U^P(\mathfrak{o}_S)$ acts properly on the quotient $P(\mathfrak{o}_S) \backslash P(F_S)w_0U^P(F_S) \cap G(\mathfrak{o}_S)$ (see Section 5.2, Corollary 5.8, below), we obtain

$$\int_{U^P(F_S)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} \sum_{\gamma \in P(\mathfrak{o}_S) \backslash P(F_S)w_0U^P(F_S) \cap G(\mathfrak{o}_S)/U^P(\mathfrak{o}_S)} f_s(\iota(\gamma)u_M u^P) \overline{\psi(u_M u^P)} du_M du^P,$$

where we have factored $U = U_M U^P$, writing $u \in U$ as $u = u_M u^P$, $u_M \in U_M$, $u^P \in U^P$.

Let U_-^P be the opposite unipotent to U^P . Next we replace double coset representatives γ by γw_0 such that

$$\gamma \in P(\mathfrak{o}_S) \backslash (P(F_S)U_-^P(F_S) \cap G(\mathfrak{o}_S))/U^P(\mathfrak{o}_S),$$

and apply the group decomposition theorem of Section 5 (see Corollary 5.8 and Proposition 5.9) together with Proposition 2.5 above. This decomposition expresses each such γ as a product of N embedded $SL_2(\mathfrak{o}_S)$ matrices according to a reduced decomposition for the long element in the relative Weyl group, where N is the dimension of U^P . The double cosets are parametrized by the bottom rows (c_j, d_j) of the $SL_2(\mathfrak{o}_S)$ matrices g_j with $d_j \neq 0$ and with (c_j, d_j) modulo units for $1 \leq j \leq N$ and c_j modulo $D_j := d_j \prod_{\ell=j+1}^N d_\ell^{\langle \gamma_j, \gamma_\ell^\vee \rangle}$. The γ_j here are the roots in U^P and the ordering of these roots is described in Section 5.

By repeated use of a rank one Bruhat decomposition, and keeping track of the contributions from the cocycles, we arrive at a decomposition $\gamma = v^P \tilde{\mathfrak{D}} w_0 u_\gamma$ for each coset representative γ where $v^P \in U^P(F_S)$, $\tilde{\mathfrak{D}} \in \tilde{G}$ projects to an element \mathfrak{D} of $T(F_S)$, and $u_\gamma \in U(F_S)$. (In the notation of Proposition 5.9, $u_\gamma = w_0 v w_0^{-1}$. As indicated in this result, u_γ depends on the choice of g_j .) We thus obtain:

$$\int_{U^P(F_S)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} \sum_{\substack{g_j \\ j=1, \dots, N}} \left[\prod_{k=1}^N (d_k, c_k)_S^{q_k} \left(\frac{d_k}{c_k} \right)^{q_k} \right] \times \\ f_s(v^P \tilde{\mathfrak{D}} w_0 u_\gamma u_M u^P) \overline{\psi(u_M u^P)} du_M du^P,$$

where the g_j have bottom row (c_j, d_j) , varying over coprime pairs with additional conditions as above, and $\tilde{\mathfrak{D}} \in \tilde{G}$ is given explicitly in Proposition 5.9 below. For $c, d \in \mathfrak{o}_S$, the symbol $(d, c)_S$ is the product of the local Hilbert symbols $(d, c)_v$ over $v \in S$. The q_k are the integers $Q(\gamma_k^\vee)$ coming from the choice of bilinear form in the definition of \tilde{G} . Moreover, the group elements v^P, u_γ depend only on this choice of g_j while the $\tilde{\mathfrak{D}}$ depends only on the d_j .

The reciprocity law (see for example Neukirch [38], Chapter VI, Theorem 8.3) implies that each of the terms in the product may be rewritten:

$$(d_k, c_k)_S \left(\frac{d_k}{c_k} \right) = \left(\frac{c_k}{d_k} \right).$$

For $\text{Re}(s)$ sufficiently large, after a variable change we then obtain

$$\sum_{\substack{g_j \\ j=1, \dots, N}} \psi(u_\gamma) \prod_{k=1}^N \left(\frac{c_k}{d_k} \right)^{q_k} \int_{U^P(F_S)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} f_s(\tilde{\mathfrak{D}} w_M w^P u_M u^P) \overline{\psi(u_M u^P)} du_M du^P$$

where we have factored $w_0 = w_M w^P$. Since $\psi((w^P)^{-1} u_M w^P) = \psi(u_M)$, this becomes

$$\sum_{\substack{g_j \\ j=1, \dots, N}} \psi(u_\gamma) \prod_{k=1}^N \left(\frac{c_k}{d_k} \right)^{q_k} \int_{U^P(F_S)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} f_s(\tilde{\mathfrak{D}} w_M u_M w^P u^P) \overline{\psi(u_M u^P)} du_M du^P. \quad (11)$$

Note further that the invariance of f_s under left multiplication by elements $P(\mathfrak{o}_S) \cap U^-(F_S)$ ensures that the character ψ must be trivial on all elements of the form $(\mathfrak{D} w_0)^{-1} p (\mathfrak{D} w_0)$ with $p \in P(\mathfrak{o}_S) \cap U^-(F_S)$. Indeed, we just move these elements p rightward in f_s and change variables. Hence, for any fixed choice of ψ , this places further divisibility conditions on the

d_i in order to yield a non-zero sum (see Lemma 6.1). It follows that the exponential sum appearing in (11) depends only on the bottom row elements (c_j, d_j) of g_j . Summing over $c_j \bmod D_j$ for fixed choices of d_j with $j = 1, \dots, N$, we denote the result

$$H(d_1, \dots, d_N) := \sum_{c_j \bmod D_j} \psi(u_\gamma) \prod_{k=1}^N \left(\frac{c_k}{d_k} \right)^{q_k}. \quad (12)$$

(The u_γ here do depend on the g_j so this is a modest abuse of notation; as noted the sum is independent of the choices of top rows.)

The integral appearing in the series (11) is given by

$$\begin{aligned} & \int_{U^P(F_S)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} f_s(\tilde{\mathfrak{D}} w_M u_M w^P u^P) \overline{\psi(u_M u^P)} du_M du^P = \\ & \delta_P^{s+1/2}(\mathfrak{D}) \int_{U^P(F_S)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} \Lambda' \circ \Lambda \circ [\pi_{\tilde{M}}(\tilde{\mathfrak{D}} w_M u_M) \cdot \varphi_s(w^P u^P)] \overline{\psi(u_M u^P)} du_M du^P. \end{aligned} \quad (13)$$

Thus (11) may be expressed as a Dirichlet series in s :

$$\sum_{\substack{d_j \in \mathfrak{o}_S / \mathfrak{o}_S^\times, d_j \neq 0 \\ j=1, \dots, N}} H(d_1, \dots, d_N) \delta_P^{s+1/2}(\mathfrak{D}) I_{f_s}(d_1, \dots, d_N),$$

where $I_{f_s}(d_1, \dots, d_N)$ is the double integral appearing on the right-hand side of (13).

To further simplify $I_{f_s}(d_1, \dots, d_N)$, write

$$\tilde{\mathfrak{D}} w_M u_M = w_M (w_M^{-1} \tilde{\mathfrak{D}} w_M) u_M = w_M u_M^{\mathfrak{D}} (w_M^{-1} \tilde{\mathfrak{D}} w_M),$$

where $u_M^{\mathfrak{D}}$ denotes u_M conjugated by $w_M^{-1} \tilde{\mathfrak{D}} w_M$. Note that $w_M^{-1} \tilde{\mathfrak{D}} w_M = (\mathfrak{D}^{w_M}, \zeta_{\mathfrak{D}})$ with $\mathfrak{D}^{w_M} \in M$ and $\zeta_{\mathfrak{D}} \in \mu_n$. Then, further factoring $\mathfrak{D}^{w_M} = (\mathfrak{D}_1, \mathfrak{D}_2)r$ with $(\mathfrak{D}_1, \mathfrak{D}_2) \in M_0 \simeq M_1^0 \times M_2^0$ and $r \in R$, we obtain

$$\tilde{\mathfrak{D}} w_M u_M = w_M u_M^{\mathfrak{D}}(\mathfrak{D}, \zeta_{\mathfrak{D}}) = w_M u_M^{\mathfrak{D}} \mathbf{s}(\mathfrak{D}_1, \mathfrak{D}_2) \mathbf{s}(r) \zeta_{\mathfrak{D}} \sigma((\mathfrak{D}_1, \mathfrak{D}_2), r)^{-1}.$$

Then $I_f(d_1, \dots, d_N)$ is equal to $\zeta_{\mathfrak{D}} \sigma((\mathfrak{D}_1, \mathfrak{D}_2), r)^{-1}$ times

$$\int_{U^P(F_S)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} [\pi_{\tilde{M}}(\mathbf{s}(r)) \cdot \varphi_s(w^P u^P)](e) (w_M u_M^{\mathfrak{D}} \mathbf{s}(\mathfrak{D}_1, \mathfrak{D}_2)) \overline{\psi(u_M u^P)} du_M du^P. \quad (14)$$

Here $[\pi_{\tilde{M}}(\mathbf{s}(r)) \cdot \varphi_s(w^P u^P)](e)$ denotes evaluation of the function $[\pi_{\tilde{M}}(\mathbf{s}(r)) \cdot \varphi_s(w^P u^P)]$ on \tilde{M} at the identity e in \tilde{M} . Thus it defines an element of $V_1 \otimes V_2$ and the inner integral is a Whittaker coefficient of an automorphic form in $V_1 \otimes V_2$.

To say more, we restrict to the test vector $f_{s,\mathfrak{a}}$ obtained from $\varphi_{s,\mathfrak{a}}$ as explained above. Then separating $w^P u^P$ into its finite and infinite components and extracting the components in $\tilde{m}_{\text{fin}}(w^P u^P) = 1$ and $m_\infty = m_\infty(w^P u^P)$ according to (8), then (14) becomes

$$\begin{aligned} & \int_{U^P(\mathfrak{a}) \times U^P(F_\infty)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} [\pi_{\tilde{M}}(\mathbf{s}(r)) \pi_s(1, m_\infty) \cdot \phi_{f_1, f_2, \Psi}](e) (w_M u_M^{\mathfrak{D}} \mathbf{s}(\mathfrak{D}_1, \mathfrak{D}_2)) \overline{\psi(u_M u^P)} du_M du^P \\ & = \mu(\mathfrak{a}) \int_{U^P(F_\infty)} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} \Psi(r) \pi_s((1, m_\infty))(f_1 \otimes f_2) (w_M u_M^{\mathfrak{D}} \mathbf{s}(\mathfrak{D}_1, \mathfrak{D}_2)) \overline{\psi(u_M u^P)} du_M du^P, \end{aligned}$$

where $\mu(\mathfrak{a})$ denotes the measure of $U^P(\mathfrak{a}) \cap U^P(\mathfrak{o}_{\text{fin}})$. Thus including the above factor of $\zeta_{\mathfrak{D}}\sigma((\mathfrak{D}_1, \mathfrak{D}_2), r)^{-1}$ into $I_{f_s}(d_1, \dots, d_N)$ and making use of (6), $I_{f_s}(d_1, \dots, d_N)$ equals

$$\mu(\mathfrak{a})\zeta_{\mathfrak{D}}\Psi(\mathfrak{D}) \int_{U^P(F_{\infty})} \int_{U_M(\mathfrak{o}_S) \backslash U_M(F_S)} \pi_s((1, m_{\infty}))(f_1 \otimes f_2)(w_M u_M^{\mathfrak{D}} \mathbf{s}(\mathfrak{D}_1, \mathfrak{D}_2)) \overline{\psi(u_M u^P)} du_M du^P.$$

Now the cover splits at the infinite places, and the Whittaker models for f_1 and f_2 at those places are thus unique. Hence, after a change of variables in U_M to undo the conjugation by \mathfrak{D}^{w_M} , the inner integral gives the Whittaker coefficients of f_1 and f_2 associated with the character $u_M \mapsto \psi(u_M^{(\mathfrak{D}^{w_M})^{-1}})$, denoted $c_{f_1, f_2}^{\psi}(\mathfrak{D})$, times the archimedean Whittaker functions of f_1 and f_2 . These are evaluated at the Borel (or $P(F_{\infty})$)-part of $w^P u^P$ written in Iwasawa coordinates. We also get a contribution of $\delta_P^{s+1/2}$ on this part. The result is a Jacquet-type integral representing the Whittaker function of $G(F_{\infty})$ attached to the representation with Langlands parameters determined by the inducing data and s . We refer to this function as $\mathcal{W}_{f_1, f_2, s}(1)$, noting in particular that it is independent of \mathfrak{D} . Collecting these results in a theorem, we have shown:

Theorem 4.1. *Given a cominuscule parabolic P , the Whittaker coefficient of the maximal parabolic Eisenstein series $E_{f, s}$ is given by*

$$\int_{U(\mathfrak{o}_S) \backslash U(F_S)} E_{f, s}(u) \overline{\psi(u)} du = \sum_{\substack{d_j \in \mathfrak{o}_S / \mathfrak{o}_S^{\times}, d_j \neq 0 \\ j=1, \dots, N}} H(d_1, \dots, d_N) \delta_P^{s+1/2}(\mathfrak{D}) I_{f_s}(d_1, \dots, d_N).$$

Here $H(d_1, \dots, d_N)$ is the exponential sum defined in (12), \mathfrak{D} is computed from the d_j as in Proposition 5.9, and $I_{f_s}(d_1, \dots, d_N)$ is defined in (13). Further, if one sets $\varphi_s = \varphi_{s, \mathfrak{a}}$ as defined in (8) with associated test vector $f := f_{s, \mathfrak{a}}$, then the Whittaker coefficient equals

$$\mathcal{W}_{f_1, f_2, s}(1) \sum_{\substack{d_j \in \mathfrak{o}_S / \mathfrak{o}_S^{\times}, d_j \neq 0 \\ j=1, \dots, N}} H(d_1, \dots, d_N) \delta_P^{s+1/2}(\mathfrak{D}) \Psi(\mathfrak{D}) \zeta_{\mathfrak{D}} c_{f_1, f_2}^{\psi}(\mathfrak{D}). \quad (15)$$

From Proposition 3.3, we deduce that the Whittaker coefficient (15), when multiplied by a suitable normalizing factor, has analytic continuation to all complex s and satisfies a functional equation in which s is replaced by $1 - s$. For the trivial cover case $n = 1$, this observation is the starting point for Langlands-Shahidi theory, but for $n > 1$ these series had not been exhibited before except in a few instances. Moreover, one may then apply Tauberian methods to obtain information about the growth of the coefficients of these series. Below, our focus will be on the combinatorial and representation-theoretic description of the exponential sum H that appears in (15), so we do not pursue this further here. However, we plan to study these consequences in subsequent works.

We remark that one could carry out this computation adelically, writing an adelic Eisenstein series as discussed following Proposition 3.3 and integrating over $U(F) \backslash U(\mathbb{A})$. Using the factorization $U = U_M U^P$ and unfolding in the usual way, one gets an iterated integral over $U^P(\mathbb{A})$ and over $U_M(F) \backslash U_M(\mathbb{A})$, the latter giving rise to the Whittaker coefficients of the inducing data. The difficulty here is that the section in the Eisenstein series is not factorizable, and the resulting expression is thus not simply a product of local terms. (In addition, the Whittaker integral does not arise from a unique model and is not factorizable.)

In fact, this difficulty arises already for the Borel Eisenstein series on an n -fold cover of GL_2 , whose Whittaker coefficients are infinite sums of n -th order Gauss sums and are not Eulerian; compare the treatments in [7], Section 5 (working over F_S) and [24], Ch. II.3 (working over \mathbb{A}). Working over F_S for arbitrarily large but finite S carries the same information as the corresponding adelic calculation.

5. DOUBLE COSET PARAMETRIZATIONS

In this section, we present an explicit parametrization for double cosets appearing in the definition of Eisenstein series. The idea is to use embedded rank one subgroups on which the metaplectic cocycle has a simple form. First, we perform the parametrization over the points of the algebraic group, from which the generalization to metaplectic covers readily follows. The use of Chevalley-Steinberg relations to explore double cosets of reductive groups is a common technique in algebraic combinatorics, especially that relating to canonical bases as in [5, 37]. Indeed we connect the results of this section to canonical bases in Section 8. But because our decomposition differs from those previously appearing in the literature, including working over a ring, we give a complete and self-contained treatment. Initially in this section, we let \mathfrak{o} be any principal ideal domain with field of fractions F .

5.1. Coset parametrizations for reductive groups. Before discussing the general coset parametrization theorems of this section, we briefly recall the necessary structure theory of reductive groups via root data, for which a basic reference is Springer [44].

Let $(X, \Phi, X^\vee, \Phi^\vee)$ be the root datum corresponding to the pair (G, T) . It comes equipped with a bilinear pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbb{Z}$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all roots $\alpha \in \Phi$ and simple reflections $s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha$ which generate the Weyl group W . Let e_α denote the isomorphism from the additive group G_a to the unique closed subgroup U_α of G such that

$$te_\alpha(x)t^{-1} = e_\alpha(\alpha(t)x) \quad (16)$$

for all $x \in F$ and $t \in T(F)$. The isomorphism e_α is chosen so that, if we set

$$w_\alpha(x) := e_\alpha(x)e_{-\alpha}(-x^{-1})e_\alpha(x) \quad \text{for any } x \in F^\times,$$

then $w_\alpha(1)$ has image s_α in W . Then for any $x \in F^\times$ and any root $\beta \in \Phi$, defining the torus element $h_\beta(x) := w_\beta(x)w_\beta(1)^{-1}$, the conjugation in (16) may be rewritten

$$h_\beta(y)e_\alpha(x)h_\beta(y)^{-1} = e_\alpha(y^{\langle \alpha, \beta^\vee \rangle} x). \quad (17)$$

Let Φ^+ denote a set of positive roots with simple roots α_i . Let s_i be shorthand for the reflections corresponding to the α_i . To any $w \in W$, and a reduced decomposition as product of simple reflections $w = s_{i_1} \dots s_{i_j}$, one has

$$\Phi_w := \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\} = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j})\}.$$

Using this description, a reduced decomposition of w gives rise to a total ordering on Φ_w . For our application to parabolic induction, let P be any parabolic subgroup of G . Write the long element $w_0 = w_M w^P$ with w_M the long element of the Levi subgroup M of P . Starting from a reduced decomposition for $w_M = s_{i_1} \dots s_{i_t}$, extend this to a reduced decomposition for $w_0 = s_{i_1} \dots s_{i_t} s_{i_{t+1}} \dots s_{i_{t+N}}$. Let $\mathbf{i} = (i_1, \dots, i_{t+N})$ be the corresponding “reduced word” and $>_{\mathbf{i}}$ be the corresponding ordering on Φ^+ so that

$$\alpha_{i_1} >_{\mathbf{i}} s_{i_1}(\alpha_{i_2}) >_{\mathbf{i}} \dots >_{\mathbf{i}} s_{i_1} \dots s_{i_{t+N-1}}(\alpha_{i_{t+N}}). \quad (18)$$

The word may then be used to give an ordered parametrization of the positive roots Φ_P in U^P , the unipotent radical of the parabolic P . We index the roots of Φ_P by γ_i as follows:

$$\gamma_1 = w_M(\alpha_{i_{t+1}}), \quad \gamma_2 = w_M s_{i_{t+1}}(\alpha_{i_{t+2}}), \quad \dots, \quad \gamma_N = w_M s_{i_{t+1}} \cdots s_{i_{t+N-1}}(\alpha_{i_{t+N}}). \quad (19)$$

Here $\gamma_j >_i \gamma_k$ if $j < k$. Note that $\gamma_N = \alpha_{i_{t+N}}$, a simple root. The group U_-^P , the opposite of the unipotent radical of P , is generated by all $e_{-\alpha}(x)$ with $\alpha \in \{\gamma_1, \dots, \gamma_N\}$.

Steinberg [45] gave a presentation for $G(F)$ valid over any field and depending only on F , the root datum, and structure constants $\eta_{\alpha,\beta;i,j}$ appearing in the commutator relation

$$e_\alpha(s)e_\beta(t)e_\alpha(s)^{-1} = e_\beta(t) \left[\prod_{\substack{i,j \in \mathbb{Z}^+ \\ i\alpha + j\beta = \gamma \in \Phi}} e_\gamma(\eta_{\alpha,\beta;i,j} s^i t^j) \right] \quad (\alpha + \beta \neq 0). \quad (20)$$

The constants $\eta_{\alpha,\beta;i,j}$ may be taken to be integers and the ordering on the product corresponds to a total ordering of the roots Φ . We take the total ordering as in (18) extended to $\Phi = \Phi^- \cup \Phi^+$ in the obvious way. Strictly speaking, Steinberg's presentation in [45] is for Chevalley groups, but see [44], Chapter 9, for a presentation valid for any reductive group.

The following result will be used repeatedly in the decomposition theorems below.

Lemma 5.1. *Fix a reduced word \mathbf{i} and ordering $<_i$ on Φ^+ . Let $\alpha, \beta \in \Phi^+$.*

- a) *Suppose $\alpha <_i \beta$ and there exists $\gamma \in \Phi$ of the form $\gamma = i\alpha - j\beta$ for some positive integers i, j . Then either $\gamma \in \Phi^+$ with $\gamma <_i \alpha$ or $\gamma \in \Phi^-$ with $-\gamma >_i \beta$.*
- b) *Suppose $\alpha >_i \beta$ and there exists $\gamma \in \Phi$ of the form $\gamma = i\alpha + j\beta$ for some positive integers i, j . Then $\alpha >_i \gamma >_i \beta$.*

Proof of (a). Write

$$\alpha = s_{i_1} \cdots s_{i_{m-1}} \alpha_{i_m}, \quad \beta = s_{i_1} \cdots s_{i_{\ell-1}} \alpha_{i_\ell} \quad \text{where } m > \ell.$$

Then setting $w = s_{i_1} \cdots s_{i_{m-1}}$, we have $w^{-1}(\alpha) \in \Phi^+$ while $w^{-1}(\beta) \in \Phi^-$ (since $\beta \in \Phi_w$). Thus $w^{-1}(\gamma) = iw^{-1}(\alpha) - jw^{-1}(\beta) \in \Phi^+$. But w is characterized by $\Phi_w = \{\delta \in \Phi^+ \mid w^{-1}(\delta) \in \Phi^-\} = \{\delta \in \Phi^+ \mid \delta >_i \alpha\}$ so if $\gamma \in \Phi^+$, then $\gamma <_i \alpha$.

Using similar arguments for $-w_0\alpha$ and $-w_0\beta$ we may show that if $\gamma \in \Phi^-$, then $-\gamma >_i \beta$. Pick $v = s_{i_{t+N}} \cdots s_{i_{\ell+1}}$ so that $v^{-1}(-w_0\alpha) \in \Phi^-$ while $v^{-1}(-w_0\beta) \in \Phi^+$. Since $\gamma \in \Phi^-$ if and only if $-w_0\gamma \in \Phi^-$, then $-w_0(-\gamma) \in \Phi^+$. Further $v^{-1}(-w_0(-\gamma)) \in \Phi^+$. Since $-w_0$ is order reversing, the claim follows. \square

Proof of (b). Write

$$\alpha = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad \beta = s_{i_1} \cdots s_{i_{k'-1}}(\alpha_{i_{k'}}), \quad k' > k.$$

Choose $w = s_{i_1} \cdots s_{i_{k-1}}$. Then $w^{-1}(\alpha)$ and $w^{-1}(\beta)$ are in Φ^+ and thus, so is $w^{-1}(\gamma)$. According to the explicit description of Φ_w , this implies $\gamma <_i \alpha$. Now consider:

$$-w_0\alpha = s_{i_{t+N}} \cdots s_{i_{k+1}}(\alpha_{i_k}), \quad -w_0\beta = s_{i_{t+N}} \cdots s_{i_{k'+1}}(\alpha_{i_{k'}}).$$

Choose $v = s_{i_{t+N}} \cdots s_{i_{k'+1}}$ so that $v^{-1}(-w_0\beta)$ and $v^{-1}(-w_0\alpha)$ are in Φ^+ . Thus $v^{-1}(-w_0\gamma) \in \Phi^+$ and so $-w_0\gamma <_i -w_0\alpha$. As $-w_0$ is order reversing, the claim follows. \square

For any $\alpha \in \Phi$, we continue to denote by ι_α the canonical morphism of group schemes from SL_2 to G corresponding to α :

$$\iota_\alpha : SL_2 \longrightarrow \langle e_\alpha, e_{-\alpha} \rangle, \quad \iota_\alpha \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e_\alpha(x), \quad \iota_\alpha \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = e_{-\alpha}(x).$$

Let B_1 be the standard rank one Borel subgroup of SL_2 and B_1^- the opposite Borel.

Theorem 5.2. *To any $u \in U_-^P(F)$, there exist $g_1, \dots, g_N \in SL_2(\mathfrak{o}) \cap B_1(F)B_1^-(F)$ such that*

$$\iota_{\gamma_1}(g_1) \cdots \iota_{\gamma_N}(g_N)u^{-1} \in B(F), \quad (21)$$

the F -points of the standard Borel subgroup of G .

Proof. Given any $u \in U_-^P(F)$, we may write

$$u^{-1} = e_{-\gamma_N}(x_N) \cdots e_{-\gamma_1}(x_1), \quad \text{with } x_i \in F.$$

Write $x_N = -c_N/d_N$ with $c_N, d_N \in \mathfrak{o}$ and $\gcd(c_N, d_N) = 1$. Then we may choose $a_N, b_N \in \mathfrak{o}$ such that $a_N d_N - b_N c_N = 1$, as \mathfrak{o} is a principal ideal domain. Letting

$$g_N = \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix} \quad \text{we have} \quad \iota_{\gamma_N}(g_N)e_{-\gamma_N}(x_N) = h_{\gamma_N}(d_N^{-1})e_{\gamma_N}(b_N d_N). \quad (22)$$

(Note that the choice of d_N is necessarily non-zero and hence $g_N \in B_1(F)B_1^-(F)$.) Thus

$$\iota_{\gamma_N}(g_N)u^{-1} = h_{\gamma_N}(d_N^{-1})e_{\gamma_N}(b_N d_N)e_{-\gamma_{N-1}}(x_{N-1}) \cdots e_{-\gamma_1}(x_1).$$

Now we move the terms $h_{\gamma_N}(d_N^{-1})e_{\gamma_N}(b_N d_N)$ rightward past all of the $e_{-\gamma_j}(x_j)$. We must show the result is of the form

$$e_{-\gamma_{N-1}}(x'_{N-1}) \cdots e_{-\gamma_1}(x'_1)b \quad \text{for some } b \in B(F), x'_j \in F.$$

More generally, if $2 \leq k \leq N$, we will show that for any $d, y, x_1, \dots, x_{k-1} \in F$,

$$h_{\gamma_k}(d)e_{\gamma_k}(y)e_{-\gamma_{k-1}}(x_{k-1}) \cdots e_{-\gamma_1}(x_1) = e_{-\gamma_{k-1}}(x'_{k-1}) \cdots e_{-\gamma_1}(x'_1)b, \quad (23)$$

for some $x'_1, \dots, x'_{k-1} \in F$ and $b \in B(F)$. Then choosing g_{k-1} according to x'_{k-1} in a similar manner to g_N above, the first statement of the theorem will follow by repeated application of (23).

The proof of (23) will follow from the relations (17) and (20), together with Lemma 5.1 which characterizes the terms appearing in relation (20) when we attempt to move the e_{γ_k} rightward past the terms $e_{-\gamma_j}$ with $\gamma_k <_i \gamma_j$.

Note the condition that $\gamma = i\alpha - j\beta$ in part (a) of Lemma 5.1 is precisely that appearing in Steinberg's commutation relation (20) for e_α and $e_{-\beta}$ with $\alpha, \beta \in \Phi^+$. Take $\alpha = \gamma_k$ for some k and $\beta = \gamma_j$ for some $j < k$. Thus $\alpha <_i \beta$. Applying the commutation relation (20) may produce elements of the form e_δ for additional positive roots δ which also must be moved rightward into $B(F)$. But part (a) of Lemma 5.1 ensures that these are smaller than α (and hence smaller than β) so we may repeatedly use the lemma for any such δ and γ_j with $j < k$. The total number of times a positive root may appear in a commutation is finite since the roots strictly decrease in the ordering with each application. All such one-dimensional unipotent subgroups associated to positive roots are, of course, contained in $B(F)$ so once they are moved rightward past the $e_{-\gamma_j}$ we need not consider them further.

If we commute an e_α (with $\alpha \in \Phi^+$) past an $e_{-\gamma_j}$ to produce a negative root γ , part (a) of the lemma also ensures that $-\gamma$ is larger than γ_j , so $\gamma = -\gamma_i$ for some $i < j$. That is, another element of the form e_γ already appears to the right of $e_{-\gamma_j}$ in the expression for u^{-1} . We would like to combine these unipotents at the same negative root together. Indeed, part (b) of the lemma ensures that we may pass $e_\gamma(x)$ to the right past $e_{-\gamma_j}(x_j), \dots, e_{-\gamma_{i+1}}(x_{i+1})$ without generating one-dimensional unipotent terms at negative roots not in this list. Then $e_\gamma(x) \cdot e_{-\gamma_i}(x_i) = e_{-\gamma_i}(x + x_i)$.

Thus to prove (23), it remains only to move the term $h_{\gamma_k}(d)$ rightward past all of the unipotent elements $e_{-\gamma_j}$ with $1 \leq j < k$ and show that the result is of the form given on the right-hand side of (23). This is immediate from (17). \square

Let $B_1(\mathfrak{o})$ denote the standard Borel subgroup of upper triangular matrices in $SL_2(\mathfrak{o})$. Recall that the cosets $B_1(\mathfrak{o}) \backslash SL_2(\mathfrak{o})$ are parametrized by their bottom rows – pairs of integers (c, d) with $\gcd(c, d) = 1$ – modulo the diagonal multiplication by units. Moreover, the pair (c, d) corresponds to a coset in the “big cell” $B_1(\mathfrak{o}) \backslash SL_2(\mathfrak{o}) \cap B_1(F)B_1^-(F)$ if and only if $d \neq 0$. Motivated by this, we introduce the notation

$$\mathcal{R} := \{(c, d) \in \mathfrak{o}^2 / \mathfrak{o}^\times \mid \gcd(c, d) = 1, d \neq 0\}.$$

Corollary 5.3. *The assignment of $g_1, \dots, g_N \in SL_2(\mathfrak{o}) \cap B_1(F)B_1^-(F)$ to any $u \in U_-^P(F)$ as in Theorem 5.2 induces a bijection*

$$\mathcal{R}^N \xrightarrow{\eta} P(F) \backslash P(F)U_P^-(F)$$

given by first completing $(c_i, d_i) \in \mathcal{R}^{(i)}$ to a matrix

$$g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL_2(\mathfrak{o}) \cap B_1(F)B_1^-(F)$$

and then composing with the map

$$(SL_2(\mathfrak{o}) \cap B_1(F)B_1^-(F))^N \xrightarrow{\varphi} P(F) \backslash P(F)U_P^-(F),$$

given in the previous theorem by $\varphi(g_1, \dots, g_N) = P(F)\iota_{\gamma_1}(g_1) \cdots \iota_{\gamma_N}(g_N)$. The map φ (and hence the bijection η) depends on the choices (a_i, b_i) for each pair (c_i, d_i) , but is a bijection of sets for any such assignment.

Proof. In the proof of the previous theorem, the set of $SL_2(\mathfrak{o})$ matrices (g_1, \dots, g_N) mapping to a given coset $P(F)u$ is determined according to the ordering as follows. Each g_i is determined by first choosing (c_i, d_i) depending only on the g_j with $j > i$. Then the choice of a_i, b_i such that $a_i d_i - b_i c_i = 1$ is free. Thus the map φ depends on the assignment of (a_i, b_i) to (c_i, d_i) but any choice is allowable.

There is an obvious bijection between \mathcal{R}^N and $P(F) \backslash P(F)U_P^-(F)$ given by mapping

$$\{(c_i, d_i)\} \in \mathcal{R}^N \mapsto P(F)e_{-\gamma_N}(x_N) \cdots e_{-\gamma_1}(x_1), \quad \text{where } x_i = \frac{c_i}{d_i} \text{ for } i = 1, \dots, n.$$

Writing

$$\eta(\{(c_i, d_i)\}_{i=1}^N) = P(F)e_{-\gamma_N}(x'_N) \cdots e_{-\gamma_1}(x'_1)$$

for some elements $x'_i \in F$, then the proof of the previous theorem shows that $x'_N = x_N$ (and so η agrees with the obvious bijection in rank one). Moreover the previous proof demonstrates that for each $i < N$, $x'_i = x_i + k$ for some constant $k \in F$ depending on g_j with $j > i$ (and the structure constants of the realization of G). Thus composing with these additive changes of variables preserves the bijection. \square

5.2. Restriction to “braidless” and “cominusculer” parabolic subgroups. Next, we give a parametrization for double cosets $P(F)\backslash P(F)U_-^P(F)/U_-^P(\mathfrak{o})$. The strategy for determining a set of representatives again uses the Steinberg commutation relations. These are defined over a field, but the following lemma will ensure that they may be used in certain cases to produce elements in the ring of integers \mathfrak{o} . Thus far, our statements have held for all reductive groups, all parabolics P , and all reduced expressions w^P for the “long element” of $W_M\backslash W$, i.e., the longest among minimal length left coset representatives. In this section, we restrict our attention to maximal parabolic subgroups P . First we require a definition, made in terms of fundamental weights $\omega_{\alpha_i} =: \omega_i$ for the weight lattice X of G .

Definition 5.4 (Littelmann, Section 3 of [27]). *A fundamental weight ω is said to be “braidless” if, for any $w \in W$, the following holds for pairs of simple roots α, β :*

$$\text{If } \langle w(\omega), \alpha^\vee \rangle > 0 \text{ and } \langle w(\omega), \beta^\vee \rangle > 0, \text{ then } \langle \beta, \alpha^\vee \rangle = 0.$$

If P_ω is the associated maximal parabolic subgroup of G , we say the parabolic is “braidless.”

In Lemma 3.1 of [27], the braidless fundamental weights are characterized as follows, using Bourbaki’s labeling of the Dynkin diagram for the root system Φ of rank r (so forks and double bonds in classical groups are closer to the node labeled ‘ r ’ rather than the one labeled ‘1’). A fundamental weight ω is braidless if and only if it is in the following collection:

- Φ is of type A ,
- Φ is of type B or C and $\omega = \omega_1$ or ω_r ,
- Φ is of type D and $\omega = \omega_1, \omega_{r-1}$ or ω_r ,
- $\Phi = E_6$ and $\omega = \omega_1$ or ω_6 or $\Phi = E_7$ and $\omega = \omega_7$,
- $\Phi = G_2$.

The name “braidless” refers to the fact that any element $w \in W_{M_\omega}\backslash W$, where M_ω is the Levi subgroup of P_ω , has a reduced decomposition which is unique up to exchange of orthogonal reflections (i.e., no braid relations are required).

Lemma 5.5. *Let G be a connected, reductive group with $\Phi(G) \neq G_2$. Then the maximal parabolic $P = P_\omega$ is braidless if and only if*

$$\langle \gamma_j, \gamma_k^\vee \rangle \geq 0 \quad \text{for all roots } \gamma_j, \gamma_k \in U^P.$$

Proof. We use the characterization of braidless weights given above, and break the proof into finitely many cases according to Cartan type. By linearity of the inner product, the question reduces to a study of inner products for adjacent simple roots. For example, in type A if α_j denotes the omitted root, then all roots in U^P are of the form $\gamma = \alpha_i + \cdots + \alpha_k$ with $i \leq j \leq k$. But $\langle \alpha_i, \alpha_i^\vee \rangle = 2$ for all simple roots α_i and $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ if $i = j \pm 1$ and $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ otherwise, so the inner product of roots in U^P is non-negative. As the remaining cases are a similar elementary verification, we omit the details.

To prove the converse, that the inner product condition on U^{P_ω} implies the weight is ω is braidless, we simply produce counterexamples. The roots in U^P are characterized as those whose expression, as a linear combination of simple roots, involves the omitted simple root α . If P is not braidless, it is straightforward to construct inner products $\langle \gamma, \alpha^\vee \rangle = -1$ with $\gamma \in U^P$. For example, in type B with omitted simple long root α_j with $j > 1$ (and necessarily with $j < n$)

$$\langle \alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_n, \alpha_j^\vee \rangle = -1.$$

We leave the remaining counterexamples, easily produced from the tables of positive roots in the appendices of Bourbaki [6], to the reader. \square

Definition 5.6. A fundamental weight ω_i is said to be “cominusculer” if its associated simple root α_i appears with coefficient 1 in the expansion of the highest root. In keeping with earlier conventions, the associated maximal parabolic $P := P_{\omega_i}$ is said to be cominusculer if ω_i is cominusculer.

A simple case-by-case check according to Cartan type shows that cominusculer weights are a subset of braidless weights.

Proposition 5.7. If $\Phi(G) \neq G_2$ and P is braidless, then there exists a bijection between the double cosets in $P(F) \backslash P(F)U_-^P(F)/U_-^P(\mathfrak{o})$ and pairs of integers (c_j, d_j) with $\gcd(c_j, d_j) = 1$, $d_j \neq 0$ and modulo units, and c_j modulo D_j , where D_j is an integer depending on the entries of $\iota(\gamma_k)$ with $k \geq j$. In particular if P is cominusculer then

$$D_j = d_j \prod_{\ell=j+1}^N d_\ell^{\langle \gamma_j, \gamma_\ell^\vee \rangle}.$$

Proof. In light of Corollary 5.3, it suffices to determine the effect of right multiplication by any element $u \in U_-^P(\mathfrak{o})$ on $\iota_{\gamma_1}(g_1) \cdots \iota_{\gamma_N}(g_N)$. Given any such u , factor it as

$$u = e_{-\gamma_1}(t_1) \cdots e_{-\gamma_N}(t_N) \quad \text{with } t_j \in \mathfrak{o}.$$

In order to perform the multiplication $\iota_{\gamma_1}(g_1) \cdots \iota_{\gamma_N}(g_N)u$, we break each $\iota_{\gamma_k}(g_k)$ into $h_{\gamma_k}, e_{\gamma_k}$, and $e_{-\gamma_k}$ as before and use the commutation relations (17) and (20) to move each $e_{-\gamma_j}(t_j)$ in u leftward until it is adjacent to $\iota_{\gamma_j}(g_j)$.

Caution is required here as the decomposition of $\iota_{\gamma_k}(g_k)$ into $h_{\gamma_k}, e_{\gamma_k}$, and $e_{-\gamma_k}$ is only defined over the field F . In particular, the commutation of $e_{-\gamma_j}(t_j)$ with $t_j \in \mathfrak{o}$ and $h_{\gamma_k}(d_k^{-1})$ for certain pairs of positive roots γ_j, γ_k in the unipotent radical of P dilates the argument t_j by an element of \mathfrak{o} only if $\langle \gamma_j, \gamma_k^\vee \rangle \geq 0$. This is guaranteed by Lemma 5.5.

The process of moving $e_{-\gamma_j}(t_j)$ leftward will produce linear changes of coordinates in t_j and create many new elements e_α for $\alpha \in \Phi$ whose arguments depend on t_j and the matrices g_i with $i \in [j+1, N]$. It suffices to compute the dilation factor appearing in the linear change $t_j \mapsto t'_j$ once $e_{-\gamma_j}(t'_j)$ is adjacent to $\iota_{\gamma_j}(g_j)$; as t_j is arbitrary in \mathfrak{o} then $c_j \in g_j$ may be restricted to a finite set determined by the size of the dilation. Upon fixing a choice of t_j , all of the e_α created by moving $e_{-\gamma_j}(t_j)$ leftward may be considered to have constant arguments in t_j (now fixed) and the entries of the g_i . As we move subsequent $e_{-\gamma_{j'}}(t_{j'})$ leftward for $j' > j$, these e_α 's involving t_j produce harmless additive changes of variables in the $t_{j'}$ which are isomorphisms of \mathfrak{o} .

To determine the required dilation factor, we need only compute the effect of moving $e_{-\gamma_j}(t_j)$ past any $\iota_{\gamma_k}(g_k)$ as in (22) with $k > j$. Then

$$\begin{aligned}
\iota_{\gamma_k}(g_k)e_{-\gamma_j}(t_j) &= h_{\gamma_k}(d_k^{-1})e_{\gamma_k}(b_k d_k)e_{-\gamma_k}(c_k/d_k)e_{-\gamma_j}(t_j) \\
&= h_{\gamma_k}(d_k^{-1})e_{\gamma_k}(b_k d_k)e_{-\gamma_j}(t_j) \prod_{\gamma=a\gamma_k+b\gamma_j \in \Phi} e_{-\gamma}(\eta_{-\gamma_k, -\gamma_j; a, b} t_j^b (c_k/d_k)^a) e_{-\gamma_k}(c_k/d_k) \\
&= e_{-\gamma_j}(d_k^{\langle \gamma_j, \gamma_k^\vee \rangle} t_j) h_{\gamma_k}(d_k^{-1}) \prod_{\gamma=a'\gamma_k-b'\gamma_j \in \Phi} e_{-\gamma}(\eta_{\gamma_k, -\gamma_j; a', b'} t_j^{b'} (b_k d_k)^{a'}) e_{\gamma_k}(b_k d_k) \times \\
&\quad \prod_{\gamma=a\gamma_k+b\gamma_j \in \Phi} e_{-\gamma}(\eta_{-\gamma_k, -\gamma_j; a, b} t_j^b (c_k/d_k)^a) e_{-\gamma_k}(c_k/d_k). \tag{24}
\end{aligned}$$

In the leftmost product of (24), which ranges over pairs (a', b') , $\gamma \neq -\gamma_j$ so it may be ignored. However, we still need to compute the commutator of $e_{\gamma_k}(b_k d_k)$ with terms in the rightmost product of (24), as these may produce $e_{-\gamma_j}$'s. This can only happen for linear combinations $a\gamma_k + b\gamma_j$ with $b = 1$ since γ_k and γ_j are linearly independent.

Thus we obtain two cases: if $a\gamma_k + \gamma_j \in \Phi$ for some positive integer a and $k > j$, then moving $e_{-\gamma_j}(t_j)$ past $\iota_{\gamma_k}(g_k)$ results in

$$e_{-\gamma_j}(d_k^{\langle \gamma_j, \gamma_k^\vee \rangle} t_j (1 + \eta_{\gamma_k, -\gamma_k - \gamma_j; 1, 1} \eta_{-\gamma_k, -\gamma_j; 1, 1} b_k^a c_k^a)).$$

If $a\gamma_k + \gamma_j \notin \Phi$ for any a , then the dilation is simply

$$e_{-\gamma_j}(d_k^{\langle \gamma_j, \gamma_k^\vee \rangle} t_j).$$

As γ_k, γ_j are positive roots in the maximal parabolic P , they both contain the simple root α omitted from P . Thus the condition $a\gamma_k + \gamma_j \in \Phi$ is impossible if P is cominuscule. Hence the simpler formula for D_j , $j = 1, \dots, N$ in these cases follows. \square

Corollary 5.8. *If $\Phi(G) \neq G_2$ and P is braidless, then the map*

$$\begin{aligned}
\tilde{\varphi} : (B_1(\mathfrak{o}) \backslash SL_2(\mathfrak{o}) \cap B_1(F) B_1^-(F))^N &\longrightarrow P(\mathfrak{o}) \backslash G(\mathfrak{o}) \cap P(F) U_-^P(F) \\
(g_1, \dots, g_N) &\longmapsto P(\mathfrak{o}) \iota_{\gamma_1}(g_1) \cdots \iota_{\gamma_N}(g_N)
\end{aligned}$$

is a bijection of sets. Moreover, $U_-^P(\mathfrak{o})$ acts properly on the right of this quotient and the double cosets $P(\mathfrak{o}) \backslash G(\mathfrak{o}) \cap P(F) U_-^P(F) / U_-^P(\mathfrak{o})$ may be parametrized by bottom rows (c_j, d_j) of the g_j with $\gcd(c_j, d_j) = 1$, $d_j \neq 0$ and modulo units, and c_j modulo D_j , with $D_j \in \mathfrak{o}$ defined in the previous proposition.

Proof. The map φ in the previous theorem factors as $\varphi = \iota \circ \tilde{\varphi}$ where

$$\iota : P(\mathfrak{o}) \backslash G(\mathfrak{o}) \cap P(F) U_-^P(F) \hookrightarrow P(F) \backslash P(F) U_-^P(F)$$

induced by the inclusion of $G(\mathfrak{o})$ in $G(F)$. The map is injective since, if $\iota(g_1) = \iota(g_2)$, then $g_1 g_2^{-1} \in P(F) \cap G(\mathfrak{o}) = P(\mathfrak{o})$. Thus $\tilde{\varphi}$ is onto since φ is onto and ι is injective. Moreover, $\tilde{\varphi}$ is injective since φ is injective. The parametrization of the double cosets follows immediately from Theorem 5.2. \square

5.3. A remark about parabolics and coset representatives. The proof of Proposition 5.7 using Chevalley-Steinberg relations for determining double coset representatives for $P(\mathfrak{o}) \backslash G(\mathfrak{o}) \cap P(F)U_-^P(F)/U_-^P(\mathfrak{o})$ forced us to assume that the parabolic P was cominuscule. In Corollary 5.3, left coset representatives for $P(F) \backslash P(F)U_-^P(F)$ with integral entries were constructed using pairs (c_i, d_i) of bottom row elements of $SL_2(\mathfrak{o})$ for $i = 1, \dots, N$. It may be possible to refine this set of representatives to provide double coset representatives indexed by integers $c_i \bmod D_i$ where D_i is a product of d_j with $j \in [1, N]$ using alternate methods.

For example, the Klingen parabolic of Sp_4 (which omits the short root α_1) is braidless, but not cominuscule. However, if one uses the theory of generalized Plücker coordinates (cf. [19]) in conjunction with the above recipe for embedded SL_2 's, then a set of representatives for the required double cosets is given in terms of the bottom row of $\varphi(g_1, g_2, g_3)$ in Sp_4 with φ as in Corollary 5.3. For the unique convex ordering, this bottom row is (in terms of the entries (a_i, b_i, c_i, d_i) of g_i):

$$(A_1, A_2, A_3, A_4) := (a_1 c_2 d_3 - c_1 c_3, c_3 d_1 - b_1 c_2 d_3, c_1 d_2 d_3, d_1 d_2 d_3)$$

and by Plücker coordinates, the double cosets required are given by $A_i \bmod A_4$ with A_4 non-zero. Clearly the condition that A_3 is mod A_4 is equivalent to taking $c_1 \bmod d_1$. Making a change of coordinates in (A_1, A_2) by left-multiplying by $\begin{pmatrix} d_1 & c_1 \\ b_1 & a_1 \end{pmatrix}$ in SL_2 , then $(A_1, A_2) \mapsto (c_2 d_3, c_3)$ and thus the double cosets may be parametrized by c_i 's modulo products of d_j 's. The example here is simplified by the fact that there are no non-trivial Plücker relations.

5.4. Group decompositions for metaplectic covers. Corollary 5.8 ensures that the method for evaluating metaplectic Whittaker functionals works for any braidless maximal parabolic P . Indeed, there is a factorization of representatives for $P(\mathfrak{o}) \backslash G(\mathfrak{o})$ in terms of embedded rank one elements which allows for simple computation of the Kubota homomorphism. Further, because this factorization yields explicit representatives for double cosets, we may perform the characteristic unfolding arguments to evaluate the Whittaker coefficient of the parabolic Eisenstein series. We do this in the context of the metaplectic group and with $\mathfrak{o} = \mathfrak{o}_S$ for a suitably chosen finite set of places S as before. Recall that $(d, c)_S$ denotes the product of the n -th order local Hilbert symbols over $v \in S$ and $\mathbf{s}(g)$ is the section $g \mapsto (g, 1)$. Finally, write q_k for value of the quadratic form $Q(\gamma_k^\vee)$ associated to the bilinear form in the definition of \tilde{G} as in Theorem 2.1.

Proposition 5.9. *Let $g_1, \dots, g_N \in SL_2(\mathfrak{o}_S)$ be of the form*

$$g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad d_i \neq 0, \quad \text{for } 1 \leq i \leq N.$$

Then there exists a decomposition of the form:

$$\mathbf{s}(\iota_{\gamma_1}(g_1)) \mathbf{s}(\iota_{\gamma_2}(g_2)) \cdots \mathbf{s}(\iota_{\gamma_N}(g_N)) = \prod_k (d_k, c_k)_S^{q_k} v^P \tilde{\mathfrak{D}} v$$

with $v^P \in U^P(F_S)$, $v \in U_-(F_S)$, and $\tilde{\mathfrak{D}} \in \tilde{T}_{F_S}$ given by

$$\tilde{\mathfrak{D}} = \mathbf{s}(h_{\gamma_1}(d_1^{-1})) \cdots \mathbf{s}(h_{\gamma_N}(d_N^{-1})).$$

Moreover, when $-\alpha_j$ is in the parabolic P , the canonical projection of $v \in U_-(F_S)$ to the unipotent subgroup $U_{-\alpha_j}$ is $e_{-\alpha_j}(v_j)$ with

$$v_j = \sum_{(k,k') \in \mathcal{S}_j} \left[(-1)^{i+i'} \eta_{i,i',k,-k'} (b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{\ell \geq k} (d_\ell^{-1})^{\langle \alpha_j, \gamma_\ell^\vee \rangle} \prod_{k' < \ell < k} (d_\ell^{-1})^{i' \langle \gamma_{k'}, \gamma_\ell^\vee \rangle} \right] \quad (25)$$

where \mathcal{S}_j denotes the set of all pairs (k, k') with $k > k'$ such that $i\gamma_k - i'\gamma_{k'} = -\alpha_j$ for positive integers i, i' . Here, $\eta_{i,i',k,-k'} := \eta_{i,i',\gamma_k, -\gamma_{k'}}$ are the integer coefficients that appear in the Steinberg commutation relation (20). The projection of v to the unipotent subgroup $U_{-\alpha_j}$ with $-\alpha_j$ not in P is equal to $e_{-\alpha_j}(c_N/d_N)$.

Remark 5.10. This result applies to all maximal parabolics P . If P is assumed braidless, any reduced decomposition for w^P is unique up to exchange of orthogonal reflections (cf. Lemma 3.2 of [27]), whose corresponding embedded rank one subgroups commute. Thus in all of the above results, the right-hand side of the factorization identity depends only on the choice of parabolic P , and not on the reduced decomposition of w^P .

Proof. The existence of the decomposition over reductive groups was proved in the previous section. In order to perform it over the metaplectic group, we need only keep track of the cocycles, which enter as follows. In decomposing each $\mathbf{s}(\iota_{\gamma_k}(g_k))$ according to a rank one Bruhat decomposition, we obtain

$$\mathbf{s}(\iota_{\gamma_k}(g_k)) = (d_k, c_k)_S^{q_k} \mathbf{s}(e_{\gamma_k}(b_k/d_k)) \mathbf{s}(h_{\gamma_k}(d_k^{-1})) \mathbf{s}(e_{-\gamma_k}(c_k/d_k)). \quad (26)$$

We move $e_{-\gamma_{k'}}$ rightward past the pieces of ι_{γ_k} where $k > k'$. If we wish to keep track of the terms v_j appearing in $v \in U_-(F_S)$, then there are three sources. First, moving the unipotent $e_{-\gamma_{k'}}(x)$ rightward past $h_{\gamma_\ell}(d_\ell^{-1})$ with $k' < \ell < k$ results in

$$e_{-\gamma_{k'}}(x d_\ell^{-\langle \gamma_{k'}, \gamma_\ell^\vee \rangle}) \quad (27)$$

according to (17). Next, we move the unipotent in (27) rightward, past $\iota_{\gamma_k}(g_k)$ for a pair (k, k') in \mathcal{S}_j and record when $i\gamma_k - i'\gamma_{k'} = -\alpha_j$ for a simple root α_j (as the associated commutation relations (20) produce a term $e_{-\alpha_j}$ whose argument includes that of (27) to the power i'). Then we move each of the unipotent elements $e_{-\alpha_j}(x)$ created by a commutation of γ_k and $\gamma_{k'}$ rightward past $h_{\gamma_\ell}(d_\ell^{-1})$ with index $\ell \geq k$, which multiplies x by $(d_\ell^{-1})^{\langle \alpha_j, \gamma_\ell^\vee \rangle}$ according to (17).

As for the resulting metaplectic cocycle, the individual Bruhat decompositions produce S -Hilbert symbols $(d_k, c_k)_S^{q_k}$ for each g_k , $k = 1, \dots, N$. According to [29] (or Section 6 of [45]) the only other nontrivial cocycles arise from multiplications of diagonal elements h_{γ_k} , so the result follows. \square

6. TWISTED MULTIPLICATIVITY

Let $\psi : F_S \longrightarrow \mathbb{C}$ be a character of F_S that is trivial on \mathfrak{o}_S but no larger fractional ideal. Given an element $\mathbf{t} = (t_1, \dots, t_r) \in (\mathfrak{o}_S - \{0\})^r$, let $\psi_{\mathbf{t}}$ be the character of $U(F_S)$ such that $\psi_{\mathbf{t}}(e_{\alpha_j}(x)) = \psi(t_j x)$ for $x \in F_S$ and $j = 1, \dots, r$.

In Theorem 4.1, we showed that for special choice of test vector, the $\psi_{\mathbf{t}}$ -Whittaker coefficient is expressible in the form

$$\mathcal{W}_{f_1, f_2, s}(1) = \sum_{\mathbf{d} \in (\mathfrak{o}_S - \{0\} / \mathfrak{o}_S^\times)^N} H(\mathbf{d}; \mathbf{t}) \delta_P^{s+1/2}(\mathfrak{D}) \Psi(\mathfrak{D}) \zeta_{\mathfrak{D}} c_{f_1, f_2}^{\psi_{\mathbf{t}}}(\mathfrak{D}). \quad (28)$$

Here we have used \mathbf{d} to denote the N -tuple (d_1, \dots, d_N) with $d_i \in \mathfrak{o}_S / \mathfrak{o}_S^\times$ and non-zero. We have also written $H(\mathbf{d}; \mathbf{t})$ in place of $H(\mathbf{d})$ to emphasize the dependence on the character $\psi_{\mathbf{t}}$. Thus explicitly, we may write $H(\mathbf{d}; \mathbf{t})$ using the definition in (12) as

$$H(\mathbf{d}; \mathbf{t}) := \sum_{c_j \pmod{D_j}} \psi\left(\sum_j t_j v_j\right) \prod_{k=1}^N \left(\frac{c_k}{d_k}\right)^{q_k} \quad (29)$$

with v_j as defined in (25) and D_j as in Proposition 5.7. Implicit here is that summands are 0 unless the following divisibility condition holds.

Lemma 6.1. *To any $\mathbf{t} = (t_1, \dots, t_r) \in (\mathfrak{o}_S - \{0\})^r$, the coefficient $H(\mathbf{d}; \mathbf{t})$ vanishes unless, for each simple root $\alpha_j \in P$,*

$$t_j \prod_{i=1}^N d_i^{-\langle \alpha_j, \gamma_i^\vee \rangle} \in \mathfrak{o}_S.$$

Proof. As noted in the discussion below (11), the Whittaker integral vanishes unless the character $\psi_{\mathbf{t}}$ is trivial on all elements of the form

$$(\mathfrak{D}w_0)^{-1} p(\mathfrak{D}w_0) \quad \text{with } p \in P(\mathfrak{o}_S) \cap U_-(F_S).$$

Here, $\mathfrak{D} = h_{\gamma_1}(d_1^{-1}) \cdots h_{\gamma_N}(d_N^{-1})$ according to Proposition 5.9. Thus it suffices to check the condition for $p = e_{-\alpha_j}(x)$ with α_j a simple root in the the parabolic P and any $x \in \mathfrak{o}_S$. The result now follows from repeated application of (17). \square

The results of Section 5 imply that the exponential sum $H(\mathbf{d}; \mathbf{t})$ has a particularly nice form when the maximal parabolic P is cominuscule. In these cases, we now demonstrate that H is multiplicative in both \mathbf{d} and \mathbf{t} up to an explicitly prescribed n -th root of unity. Collectively, these two properties are commonly referred to as “twisted multiplicativity.”

Let ω_i^\vee denote the i -th fundamental coweight, so that

$$\langle \alpha_j, \omega_i^\vee \rangle = \delta_{i,j}.$$

Proposition 6.2. *Let P be a cominuscule parabolic. Given any $\mathbf{d} = (d_1, \dots, d_N)$ with $d_i \in \mathfrak{o}_S / \mathfrak{o}_S^\times$ and non-zero, write $\mathbf{t} = (t_1 t'_1, \dots, t_r t'_r)$ such that $\gcd(\prod_{i=1}^r t_i, \prod_{j=1}^N d_j) = 1$. Then*

$$H(\mathbf{d}; \mathbf{t}) = \prod_{k=1}^N \prod_{i=1}^r \left(\frac{t_i^{-\langle \gamma_k, \omega_i^\vee \rangle}}{d_k} \right)^{q_k} H(\mathbf{d}; t'_1, \dots, t'_r).$$

Proof. According to (29), if $\mathbf{t} = (t_1 t'_1, \dots, t_r t'_r)$, then

$$H(\mathbf{d}; \mathbf{t}) = \sum_{c_i} \left[\prod_{i=1}^N \left(\frac{c_i}{d_i} \right)^{q_i} \right] \psi \left(\sum_{j=1}^r t_j t'_j v_j \right) \quad (30)$$

where, because P is assumed cominusculé, the sum is over c_i modulo

$$D_i := d_i \prod_{\ell=i+1}^N d_\ell^{\langle \gamma_i, \gamma_\ell^\vee \rangle} \quad (31)$$

for $i = 1, \dots, N$. Moreover, $v_j \in F_S$ with $j = 1, \dots, r$ is the element appearing in (25).

The result will follow by applying a change of variables

$$c_i \mapsto c_i \prod_{j=1}^r t_j^{-\langle \gamma_i, \omega_j^\vee \rangle}.$$

Indeed, the change produces the residue symbols in the statement of the proposition and is an automorphism of residue classes mod D_i according to the assumed relative primality of t_j 's and d_i 's. So it suffices to show this change of variables eliminates the dependence on t_i appearing in the argument of the character ψ in (30).

As b_k is a multiplicative inverse of $c_k \pmod{d_k \prod_{\ell=k+1}^N d_\ell^{\langle \gamma_k, \gamma_\ell \rangle}}$, then to any index j_0 and pairs (k, k') in \mathcal{S}_{j_0} (whose definition is given in Proposition 5.9):

$$b_k^i c_{k'}^{i'} \mapsto b_k^i c_{k'}^{i'} \prod_{j=1}^r t_j^{-\langle -\gamma_k \cdot i + \gamma_{k'}' \cdot i', \omega_j^\vee \rangle} = b_k^i c_{k'}^{i'} t_{j_0}^{-1}$$

where the last equality simply follows from the definition of \mathcal{S}_{j_0} . This cancels the t_{j_0} appearing in ψ for each $j_0 \in [1, \ell]$ and the result follows. \square

Twisted multiplicativity for $H(\mathbf{d}; \mathbf{t})$ with respect to \mathbf{d} will follow from the Chinese remainder theorem, provided one can demonstrate that the moduli D_i for integers c_i in the exponential sum satisfy a precise relationship with the conductors of the additive characters in the sum. This relationship is the content of the following result.

Lemma 6.3. *For each simple root α_j in the maximal parabolic P , and a pair $(k, k') \in \mathcal{S}_j$ (i.e., with $k > k'$ and $i\gamma_k - i'\gamma_{k'} = -\alpha_j$ for positive integers i, i'), set*

$$D(k, k'; \alpha_j) := d_k^i d_{k'}^{i'} \prod_{\ell \geq k} d_\ell^{\langle \alpha_j, \gamma_\ell^\vee \rangle} \prod_{k' < \ell < k} d_\ell^{i' \langle \gamma_{k'}, \gamma_\ell^\vee \rangle},$$

the product of d 's appearing in the denominator of the summand of v_j corresponding to the pair (k, k') as in (25). Then

$$D(k, k'; \alpha_j) = \frac{D_{k'}^{i'}}{D_k^i},$$

where

$$D_k := d_k \prod_{\ell=k+1}^N d_\ell^{\langle \gamma_k, \gamma_\ell^\vee \rangle},$$

the modulus of c_k ($k = 1, \dots, N$) in the exponential sum $H(\mathbf{d}; \mathbf{t})$.

Proof. This follows by straightforward calculation of $D_{k'}^{i'}/D_k^i$. The only term that requires special care is d_k , where $D_{k'}^{i'}$ contributes $d_k^{i' \langle \gamma_{k'}, \gamma_k^\vee \rangle}$ and D_k^i contributes d_k^i . But

$$i' \langle \gamma_{k'}, \gamma_k^\vee \rangle - i = \langle \alpha_j + i\gamma_k, \gamma_k^\vee \rangle - i = \langle \alpha_j, \gamma_k^\vee \rangle + i,$$

since $\langle \alpha, \alpha^\vee \rangle = 2$ for all roots α . This matches the power of d_k appearing in $D(k, k'; \alpha_j)$. \square

The result is independent of any restrictions on the parabolic P (e.g., cominuscle), so it could be used, for example, to prove twisted multiplicativity statements in the case discussed in Section 5.3.

Proposition 6.4. *Let P be cominuscle with roots $\gamma_k \in U^P$ as above. If, for $k > k'$, there exist positive integers i, i' such that*

$$i\gamma_k - i'\gamma_{k'} = -\alpha_j, \quad \text{for some simple root } \alpha_j \in P,$$

then $i = i' = 1$.

Proof. Express γ_k and $\gamma_{k'}$ as linear combinations of simple roots:

$$\gamma_k = \sum_{\ell} c_{\ell} \alpha_{\ell}, \quad \gamma_{k'} = \sum_{\ell} c'_{\ell} \alpha_{\ell}.$$

As both $\gamma_k, \gamma_{k'}$ are in U^P with P cominuscle, both sums contain the omitted root α with coefficient 1. As $\alpha \neq \alpha_j$ this forces $i = i'$. But then $i = i' = 1$, else there is no pair of positive integers c_j, c'_j such that $i(c_j - c'_j) = -1$. \square

With these results in hand, we are ready to prove the twisted multiplicativity of the exponential sum H with respect to \mathbf{d} .

Theorem 6.5. *Let P be a cominuscle parabolic. Given $\mathbf{d} = (d_1, \dots, d_N)$ with each $d_i = e_i f_i$ such that $\gcd(e_1 \cdots e_N, f_1 \cdots f_N) = 1$, define*

$$E_k := e_k \prod_{\ell=k+1}^N e_{\ell}^{\langle \gamma_k, \gamma_{\ell}^{\vee} \rangle}, \quad F_k := f_k \prod_{\ell=k+1}^N f_{\ell}^{\langle \gamma_k, \gamma_{\ell}^{\vee} \rangle}$$

in analogy with D_k in (31). Then the exponential sum $H(\mathbf{d}; \mathbf{t})$ factors as:

$$H(\mathbf{d}; \mathbf{t}) = \left(\frac{E_k}{f_k} \right)^{q_k} \left(\frac{F_k}{e_k} \right)^{q_k} H(e_1, \dots, e_N; \mathbf{t}) H(f_1, \dots, f_N; \mathbf{t}). \quad (32)$$

Proof. The result follows by use of the Chinese remainder theorem. Recall that c_k runs mod D_k as in (31). Let us define E_k and F_k similarly, with d_{ℓ} 's replaced by e_{ℓ} 's and f_{ℓ} 's, respectively, in (31) so that $D_k = E_k F_k$. Thus the c_k may be reparametrized writing

$$c_k = x_k E_k + y_k F_k \quad \text{with } x_k \pmod{F_k} \text{ and } y_k \pmod{E_k}. \quad (33)$$

Recall that $d_k || D_k$ and hence by definition $e_k || E_k$ and $f_k || F_k$, so that the residue symbols appearing in $H(\mathbf{d}; \mathbf{t})$ may be rewritten:

$$\left(\frac{c_k}{d_k} \right) = \left(\frac{x_k E_k + y_k F_k}{e_k f_k} \right) = \left(\frac{x_k E_k}{f_k} \right) \left(\frac{y_k F_k}{e_k} \right) = \left[\left(\frac{E_k}{f_k} \right) \left(\frac{F_k}{e_k} \right) \right] \left(\frac{x_k}{f_k} \right) \left(\frac{y_k}{e_k} \right).$$

Thus the twisted multiplicativity in (32) will follow if the additive character $\psi \left(\sum_{j=1}^r t_j v_j \right)$ appearing in H factors neatly into a product of characters for any choice of c_k 's in the sum H . There are two cases to consider.

Case 1: If j is the index of the omitted simple root, then u_j is simply c_N/d_N . And

$$\psi \left(t_j \frac{c_N}{d_N} \right) = \psi \left(t_j \frac{x_N E_N + y_N F_N}{e_N f_N} \right) = \psi \left(t_j \frac{x_N}{f_N} \right) \psi \left(t_j \frac{y_N}{e_N} \right),$$

where we have used that $E_N = e_N$ and $F_N = f_N$ in the last equality, as the products in the definition of E_N and F_N are empty.

Case 2: If j is not the index of the omitted simple root, recall that v_j is as in (25) and each summand in v_j is (up to an inconsequential integral structure constant $\pm\eta$) of form $b_k^i c_{k'}^{i'}/D(k, k'; \alpha_j)$ for a pair of indices $(k, k') \in \mathcal{S}_j$. Moreover since P is cominusculé, by Proposition 6.4, $i = i' = 1$ for each pair $(k, k') \in \mathcal{S}_j$. In $H(\mathbf{d}; \mathbf{t})$, the c_k are then summed mod D_k and the b_k are multiplicative inverses of the c_k . Thus to prove the theorem, it suffices to show for each j and each pair $(k, k') \in \mathcal{S}_j$:

$$\psi\left(t_j \frac{b_k c_{k'}}{D(k, k'; \alpha_j)}\right) = \psi\left(t_j \frac{\bar{x}_k x_{k'}}{F(k, k'; \alpha_j)}\right) \psi\left(t_j \frac{\bar{y}_k y_{k'}}{E(k, k'; \alpha_j)}\right), \quad (34)$$

where c_k is parametrized as in (33), \bar{x}_k and \bar{y}_k are inverses of the x_k and y_k mod F_k and E_k , respectively, and $E(k, k'; \alpha_j)$ and $F(k, k'; \alpha_j)$ are defined analogously to $D(k, k'; \alpha_j)$ with d_j 's replaced by e_j 's and f_j 's, respectively.

To prove (34), expand:

$$\frac{b_k c_{k'}}{D(k, k'; \alpha_j)} = \frac{b_k (x_{k'} E_{k'} + y_{k'} F_{k'})}{E(k, k'; \alpha_j) F(k, k'; \alpha_j)} = \frac{b_k x_{k'} E_{k'}}{F(k, k'; \alpha_j)} + \frac{b_k y_{k'} F_{k'}}{E(k, k'; \alpha_j)},$$

where we have used Lemma 6.3 with $(i, i') = (1, 1)$ in the second equality. As b_k is defined mod $E_k F_k$ with $b_k x_k E_k \equiv 1 \pmod{F_k}$, then we may rewrite

$$\frac{b_k x_{k'} E_{k'}^i}{F(k, k'; \alpha_j)} = \frac{\bar{x}_k x_{k'}}{F(k, k'; \alpha_j)},$$

so the rational expression involving $x_{k'}$ gives one of the desired factors on the right-hand side of (34). Similarly, the expression involving $y_{k'}$ gives the other factor. \square

These two twisted multiplicativity properties of the exponential sum $H(\mathbf{d}; \mathbf{t})$ appearing in (28) imply that it suffices to determine H when the components of \mathbf{d} and \mathbf{t} are powers of any fixed prime $p \in \mathfrak{o}_S$. Given such a prime and non-negative integer tuples $\boldsymbol{\ell} = (\ell_1, \dots, \ell_N)$ and $\mathbf{m} = (m_1, \dots, m_r)$, define

$$S_{\boldsymbol{\ell}, \mathbf{m}} := H(p^{\ell_1}, \dots, p^{\ell_N}; p^{m_1}, \dots, p^{m_r}). \quad (35)$$

Before making general remarks about the sums $S_{\boldsymbol{\ell}, \mathbf{m}}$ in Sections 8 and 9, we discuss an example to be used throughout the remainder of the paper.

7. AN EXAMPLE IN \widetilde{GL}_4

We now write the exponential sum explicitly in one of the first non-minimal cases: $\widetilde{G} = \widetilde{GL}_4$ and $P = MU$ with $M = GL_2 \times GL_2$. The metaplectic cover \widetilde{G} is chosen to satisfy the conditions of Theorem 2.1 with $Q(\alpha^\vee) = 1$ for all roots α . This choice does not uniquely determine \widetilde{G} (see for example Chapter 0 of [24]) but the resulting exponential sum is independent of this choice. Lastly, the structure constants $\eta_{\alpha, \beta; i, j}$ in (20) used for the realization of GL_4 may be taken to be ± 1 , as usual. The example presented here will be used repeatedly to illustrate more general phenomena discussed in the subsequent sections.

In order to apply the group decomposition theorems from Section 5, we choose a reduced expression for w_0 respecting the factorization $w_0 = w_M w^P$, where w_M is the long element

for the Levi subgroup M . Thus we may take $w_0 = s_1 s_3 s_2 s_1 s_3 s_2$ with $w^P = s_2 s_1 s_3 s_2$. This gives rise to the ordering of positive roots in U^P which are indexed according to (19) by

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2),$$

where, as usual, α_j denotes the simple positive root at position $(j, j+1)$ in GL_4 .

Given $t_1, t_2, t_3 \in \mathfrak{o}_S$ nonzero, the Whittaker coefficient in (28) has the form

$$\mathcal{W}_{f_1, f_2, s}(1) \sum_{\substack{d_1, d_2, d_3, d_4 \in \mathfrak{o}_S / \mathfrak{o}_S^\times \\ d_j \neq 0}} (d_2, d_1)_S (d_2 d_4, d_3)_S H(\mathbf{d}; \mathbf{t}) \Psi(\mathfrak{D}) c_{f_1, f_2}^{\psi_{\mathbf{t}}}(\mathfrak{D}) |d_1 d_2 d_3 d_4|^{-(1+2s)}.$$

Here

$$H(\mathbf{d}; \mathbf{t}) = \sum_{c_1, c_2, c_3, c_4} \left[\prod_{k=1}^4 \left(\frac{c_k}{d_k} \right) \right] \psi \left(-t_1 \left(\frac{b_2 c_1 d_4}{d_1 d_3} + \frac{b_4 c_3}{d_3} \right) + t_2 \frac{c_4}{d_4} + t_3 \left(\frac{c_1 b_3 d_4}{d_1 d_2} + \frac{c_2 b_4}{d_2} \right) \right) \quad (36)$$

with the sum over

$$c_1 \pmod{d_1 d_2 d_3}, \quad c_2 \pmod{d_2 d_4}, \quad c_3 \pmod{d_3 d_4}, \quad c_4 \pmod{d_4},$$

and the b_i are multiplicative inverses of c_i for their respective moduli. This sum arises only when the divisibility conditions of Lemma 6.1 are satisfied; in this case they are

$$d_1 d_3 \mid t_1 d_2 d_4, \quad d_1 d_2 \mid t_3 d_3 d_4. \quad (37)$$

We assume them henceforth.

The sum satisfies the twisted multiplicativity conditions of the previous section and so it suffices to compute the value of $H(\mathbf{d}; \mathbf{t})$ when the parameters d_i and t_i are powers of a fixed prime p in \mathfrak{o}_S . Let q denote the cardinality of $\mathfrak{o}_S / p\mathfrak{o}_S$.

Let $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$ and $\mathbf{m} = (m_1, m_2, m_3)$ be vectors of non-negative integers. Then in the notation of (35),

$$S_{\ell, \mathbf{m}} = H((p^{\ell_1}, p^{\ell_2}, p^{\ell_3}, p^{\ell_4}); (p^{m_1}, p^{m_2}, p^{m_3}))$$

where

$$S_{\ell, \mathbf{m}} := q^{2\ell_4} \sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(-p^{m_1} \left(\frac{b_2 c_1 p^{\ell_4}}{p^{\ell_1 + \ell_3}} + \frac{b_4 c_3}{p^{\ell_3}} \right) + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \left(\frac{c_1 b_3 p^{\ell_4}}{p^{\ell_1 + \ell_2}} + \frac{c_2 b_4}{p^{\ell_2}} \right) \right). \quad (38)$$

Here we sum over c_i modulo p^{ℓ_i} for $i = 2, 3, 4$ and c_1 modulo $p^{\ell_1 + \ell_2 + \ell_3}$, with c_i prime to p if $\ell_i > 0$ and no such condition if $\ell_i = 0$; b_i satisfies the congruence $b_i c_i \equiv 1 \pmod{p^{\ell_i}}$ for $i = 2, 3, 4$. The divisibility conditions become

$$\ell_1 + \ell_3 \leq m_1 + \ell_2 + \ell_4 \quad (39)$$

$$\ell_1 + \ell_2 \leq m_3 + \ell_3 + \ell_4. \quad (40)$$

Strictly speaking, (36) has c_j modulo $p^{\ell_j + \ell_4}$ for $j = 2, 3$ but using the above divisibility conditions we see that the sum depends only on c_j modulo p^{ℓ_j} for $j = 2, 3$, so the support is unchanged by viewing c_j modulo p^{ℓ_j} . Moreover, one could begin with the original sum with c_j modulo $p^{\ell_j + \ell_4}$ for $j = 2, 3$ and observe that the sum is zero unless the divisibility conditions hold. For example, if $\ell_4 > 0$ then changing c_2 to $c_2 + ap^{\ell_2}$ where a runs modulo

p^{ℓ_4} shows that the sum is zero unless $\ell_1 + \ell_3 \leq m_1 + \ell_2 + \ell_4$. Similarly, changing c_3 to $c_3 + ap^{\ell_3}$ gives the divisibility condition $\ell_1 + \ell_2 \leq m_3 + \ell_3 + \ell_4$. If instead $\ell_4 = 0$ then we may take $c_4 = b_4 = 0$ and the desired inequalities are easy to obtain.

8. CANONICAL BASES AND THE EXPONENTIAL SUM

We return to the case of arbitrary (i.e. not necessarily cominuscle) parabolics P and a reduced word \mathbf{i}^P for w^P . Recall that for a fixed prime $p \in \mathfrak{o}_S$, in (35) we set

$$S_{\ell, \mathbf{m}} := H(p^{\ell_1}, \dots, p^{\ell_N}; p^{m_1}, \dots, p^{m_r}),$$

with H the exponential sum appearing in the Whittaker coefficient as in (28). Two basic questions immediately arise:

- (1) For which $\ell \in \mathbb{Z}_{\geq 0}^N$ is $S_{\ell, \mathbf{m}} \neq 0$?
- (2) Can we give an evaluation of $S_{\ell, \mathbf{m}}$, for any choice of ℓ and \mathbf{m} , in terms of representation theoretic data on the Langlands dual group (independent of p)?

In Section 8.2, the integers ℓ are connected to the \mathbf{i}^P -Lusztig data for canonical basis elements on the dual group G^\vee corresponding the choice of reduced word \mathbf{i}^P . (This terminology is recalled in Section 8.1.) With this connection, answers to both of these questions may be formulated in terms of representation theory and canonical bases for the dual group. Roughly, we expect the answer to the first question is that the set of ℓ 's for which $S_{\ell, \mathbf{m}} \neq 0$ is *almost* the set of Lusztig data for the highest weight representation of $G^\vee(\mathbb{C})$ of highest weight $\mathbf{m} + \rho$, identifying integer r -tuples \mathbf{m} with elements of the weight lattice, with ρ the Weyl vector of the dual group G^\vee . Here “almost” means that the statement is true up to a set of ℓ 's lying on a hyperplane of strictly smaller dimension than N . For the second question, we expect the general answer is expressed using the other most important parametrization of canonical bases – Kashiwara’s string data (whose definition is reviewed in Section 8.3). Indeed, for almost all ℓ , we expect that $S_{\ell, \mathbf{m}}$ may be expressed as products of N n -th order Gauss sums whose moduli are given by the Kashiwara string data of the corresponding canonical basis elements for the G^\vee module with highest weight $\mathbf{m} + \rho$. Again “almost all” means that modifications must be made at certain degenerate points of the string polytope. Precise formulations of these results are given in the subsequent sections of the paper.

8.1. The \mathbf{i} -Lusztig data. Let $\mathcal{U} := \mathcal{U}_q(\mathfrak{g})$ be the quantized universal enveloping algebra associated to a semisimple Lie algebra \mathfrak{g} by Drinfeld and Jimbo. The algebra \mathcal{U} has a presentation in terms of generators E_j , K_j^\pm , and F_j for $j = 1, \dots, r$ where r is the rank of \mathfrak{g} . See for example [28], Section 3.1.1, (a)–(d), for their relations. Let \mathcal{U}^+ denote the subalgebra generated by the E_j .

Lusztig describes bases for \mathcal{U}^+ in terms of certain algebra automorphisms T_j on $\mathcal{U}_q(\mathfrak{g})$ for $j = 1, \dots, r$. Their precise definition is given in [28], Section 5.2 and Chapter 37, where they are called $T'_{j,-1}$. The T_j satisfy the braid relations, so extend to an action of the braid group of \mathcal{U} . Moreover, if $s_{i_1} \cdots s_{i_k}$ is a reduced expression in W , the Weyl group of \mathfrak{g} , satisfying

$$s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_j, \quad \alpha_j : \text{simple root},$$

then

$$T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(E_{i_k}) = E_j.$$

Let $\mathbf{i} = (i_1, \dots, i_N)$ be a word such that $s_{i_1} \cdots s_{i_N}$ is a reduced expression for the long element $w_0 \in W$. To each such \mathbf{i} , Lusztig associates a $\mathbb{C}(q)$ -basis $\mathcal{B}_{\mathbf{i}}$ of \mathcal{U}^+ consisting of elements

$$\left\{ p_{\mathbf{i}}^{\mathbf{c}} := E_{i_1}^{(c_1)} T_{i_1} (E_{i_2}^{(c_2)}) \cdots (T_{i_1} T_{i_2} \cdots T_{i_{N-1}}) (E_{i_N}^{(c_N)}) \mid \mathbf{c} \in \mathbb{Z}_{\geq 0}^N \right\}. \quad (41)$$

See Chapter 40 of [28] for a proof that these elements lie in \mathcal{U}^+ and form a basis.

One means of describing the canonical basis \mathcal{B} of \mathcal{U}^+ is via a graph structure on the set of pairs (\mathbf{i}, \mathbf{c}) as \mathbf{i} ranges over all reduced decompositions of the long element w_0 and \mathbf{c} runs over $\mathbb{Z}_{\geq 0}^N$ corresponding to elements of $\mathcal{B}_{\mathbf{i}}$ as in (41). This is done in two steps. First, a preliminary graph structure may be placed on the set of all reduced decompositions. The words \mathbf{i} and \mathbf{i}' are joined by an edge if the two words differ by a single application of the braid relation $s_i s_j \dots = s_j s_i \dots$.

For any pair of words \mathbf{i} and \mathbf{i}' joined in this graph, define a map $R_{\mathbf{i}}^{\mathbf{i}'} : \mathbb{N}^N \rightarrow \mathbb{N}^N$ taking $\mathbf{c} \mapsto \mathbf{c}'$ according to the braid relation between \mathbf{i} and \mathbf{i}' . If $s_i s_j = s_j s_i$ is the required braid relation and (a, b) and (a', b') are the respective consecutive entries of \mathbf{c} and \mathbf{c}' at which \mathbf{i} and \mathbf{i}' differ, then $R_{\mathbf{i}}^{\mathbf{i}'}$ restricted to these entries is

$$(a', b') = (b, a). \quad (42)$$

If instead $s_i s_j s_i = s_j s_i s_j$ is the required relation, let (a, b, c) and (a', b', c') be the respective consecutive entries of \mathbf{c} and \mathbf{c}' at which \mathbf{i} and \mathbf{i}' differ. Then $R_{\mathbf{i}}^{\mathbf{i}'}$ restricted to these entries is

$$(a', b', c') = (b + c - \min(a, c), \min(a, c), a + b - \min(a, c)). \quad (43)$$

For the maps $R_{\mathbf{i}}^{\mathbf{i}'}$ corresponding to braid relations of type B_2 and G_2 , see Section 3 of [4] or Section 7 of [32]. For each, one can check that $R_{\mathbf{i}}^{\mathbf{i}'}$ is a bijection with inverse $R_{\mathbf{i}'}^{\mathbf{i}}$. The final graph structure is given by connecting (\mathbf{i}, \mathbf{c}) to $(\mathbf{i}', \mathbf{c}')$ by an edge if \mathbf{i} and \mathbf{i}' are adjacent in the preliminary graph, and $R_{\mathbf{i}}^{\mathbf{i}'}(\mathbf{c}) = \mathbf{c}'$. Let \mathbf{X} denote the set of connected components of the resulting graph.

Theorem 8.1 (Lusztig, [28], Ch. 42). *The connected components \mathbf{X} are in canonical bijection with \mathcal{B} . Moreover, for any \mathbf{i} , the map from $\mathbb{Z}_{\geq 0}^N$ to \mathbf{X} sending elements \mathbf{c} to the connected component of (\mathbf{i}, \mathbf{c}) is a bijection.*

The vectors \mathbf{c} appearing in the above theorem are referred to as the “ \mathbf{i} -Lusztig data” for \mathcal{U}^+ . Let \mathcal{B}^\vee be the canonical basis with respect to the upper triangular part of the quantized universal enveloping algebra of G^\vee , the dual group of G .

8.2. Lusztig data, MV polytopes, and $S_{\ell, m}$. In this section, we explain the connection between the \mathbf{i} -Lusztig data and the valuations ℓ_i of the d_i appearing in $H(\mathbf{d}; \mathbf{t})$ of (28), using the theory of MV polytopes. Throughout this section, let N continue to denote the length of the reduced word \mathbf{i} . Our first goal is to define terms in and explain consequences of the following result.

Theorem 8.2 (Kamnitzer, [23] Thms. 7.1 and 7.2). *There is a coweight preserving bijection between the set of stable MV polytopes and the canonical basis \mathcal{B}^\vee . Under this bijection, the \mathbf{i} -Lusztig data of b in \mathcal{B}^\vee is equal to the integer N -tuple of edge lengths n_\bullet in the one-skeleton of the MV polytope at edges corresponding to \mathbf{i} .*

To expand on this recall that in Section 4 of [23], to a set of positive integers n_\bullet , the stable MV polytopes are (X^\vee -orbits of Zariski closures of) certain subsets $A^{\mathbf{i}}(n_\bullet)$ of the affine Grassmannian $\mathcal{G} := G(\mathbb{C}[[t]]) \backslash G(\mathbb{C}((t)))$. Thus the aim of this section is twofold:

- (1) Define the sets $A^{\mathbf{i}}(n_\bullet)$ presented in [23]. For simplicity, we define them for reduced words \mathbf{i} for the long element w_0 rather than w^P for a maximal parabolic P . It is straightforward to adapt this to a relative version whenever the word \mathbf{i} is compatible with the factorization $w_0 = w_M w^P$.
- (2) Show that the computation of n_\bullet corresponding to a unipotent element u in \mathcal{G} is formally identical to the computation of the valuations of the d_i corresponding to a point u in the big cell of the flag variety $P(\mathfrak{o}_S) \backslash G(\mathfrak{o}_S)$ in Corollary 5.8.

In proving Statement (2), we must relate an Iwasawa decomposition for an element in \mathcal{G} to a Bruhat decomposition for elements in the flag variety in Section 5; nevertheless, as we will show, the Steinberg commutation relations required in each are identical, and hence the result will follow. In view of Theorem 8.2, Statement (2) then implies that the valuations of the d_i corresponding to a reduced word \mathbf{i} are equal to the \mathbf{i} -Lusztig data (see Theorem 8.3 for the precise statement).

We now begin a systematic explanation of the terms in Theorem 8.2. The GGMS stratification of the affine Grassmannian \mathcal{G} is given by the intersections of semi-infinite cells

$$A(\lambda_\bullet) := \bigcap_{w \in W} S_w^{\lambda_w} \quad \text{where } S_w^\mu := t^\mu w U w^{-1},$$

and $\lambda_\bullet = (\lambda_w)_{w \in W}$ ranges over all N -tuples of coweights. The intersection is empty unless $w^{-1}\lambda_v \geq w^{-1}\lambda_w$ for all $v, w \in W$. Here $\mu \geq \nu$ means that $\mu - \nu$ is a non-negative linear combination of fundamental weights. See Section 5 of [1] or Section 2.4 of [23] for details.

To any such collection of coweights λ_\bullet satisfying the inequalities, define the corresponding pseudo-Weyl polytope $P(\lambda_\bullet)$ to be the intersection of cones:

$$P(\lambda_\bullet) := \bigcap_{w \in W} C_w^{\lambda_w}, \quad \text{where } C_w^{\lambda_w} := \{\alpha \in \mathfrak{t}_\mathbb{R} : w^{-1}(\alpha) \geq w^{-1}(\lambda_w)\}.$$

Here $\mathfrak{t}_\mathbb{R}$ denotes the real points of the Cartan subalgebra of \mathfrak{g} . Further define the “length” of edges in the one-skeleton of $P(\lambda_\bullet)$ as follows. Any adjacent vertices in the polytope have Weyl group elements which differ by a simple reflection s_i . Then the length n is given by

$$\lambda_{ws_i} - \lambda_w = n \cdot w(\alpha_i^\vee). \tag{44}$$

Thus given a point L in the affine Grassmannian \mathcal{G} , one may first determine the stratum $A(\lambda_\bullet)$ to which it belongs, compute the polytope $P(L) := P(\lambda_\bullet)$ corresponding to L and extract the lengths n_\bullet of the one-skeleton. Given a reduced word $\mathbf{i} = i_1 \cdots i_N$ for w_0 , we focus attention on the one-skeleton lengths for the edges between consecutive vertices in the set

$$\lambda_\bullet^{\mathbf{i}} := \{\lambda_e, \lambda_{s_{i_N}}, \lambda_{s_{i_N} s_{i_{N-1}}}, \dots, \lambda_{w_0}\}. \tag{45}$$

These are the edges corresponding to \mathbf{i} referred to in Theorem 8.2. With a view toward the theorem, Kamnitzer calls these edge lengths the “ \mathbf{i} -Lusztig data for the pseudo-Weyl polytope.” Kamnitzer’s ordering is derived from a left-to-right reading of the word. We have chosen a right-to-left reading in (45) to match the notation of Section 5.

MV polytopes arise by executing this procedure for points of \mathcal{G} in subsets of the form

$$X(\lambda) := U \cap t^\lambda U^- = S_e^0 \cap S_{w_0}^\lambda \subset \mathcal{G}$$

for dominant coweights λ . Indeed, let \mathbf{i} be a reduced word for w_0 and let n_\bullet be a set of N natural numbers. Then let $\mathcal{Q}^{\mathbf{i}}(n_\bullet; \lambda)$ denote the set of all pseudo-Weyl polytopes with one-skeleton lengths n_\bullet for edges between $\lambda_\bullet^{\mathbf{i}}$, with $\lambda_e = 0$ and $\lambda_{w_0} = \lambda$ for the fixed dominant coweight λ . Then

$$A^{\mathbf{i}}(n_\bullet) := \{u \in X(\lambda) : P(u) \in \mathcal{Q}^{\mathbf{i}}(n_\bullet; \lambda)\}.$$

In Section 4 of [23], Kamnitzer proves that the closures of the $A^{\mathbf{i}}(n_\bullet)$ are precisely the MV cycles. MV polytopes are the polytopes arising from distinguished elements of $A^{\mathbf{i}}(n_\bullet)$, but we are only concerned with the one-skeleton lengths for edges between $\lambda_\bullet^{\mathbf{i}}$ which are the same for any member of $A^{\mathbf{i}}(n_\bullet)$ by definition, and hence the lengths between $\lambda_\bullet^{\mathbf{i}}$ for the MV polytope.

Finally, stable MV polytopes are the class of MV polytopes obtained from the X^\vee orbits of MV cycles induced from the action $\mu \in X^\vee : L \mapsto L \cdot t^\mu$ on \mathcal{G} , which then acts on the polytope by translation and so preserves the one-skeleton length data n_\bullet . With this, we've completed the explanation of the objects in Theorem 8.2.

In the remainder of the section, we show that an algorithm for computing the integers n_\bullet for an element $u \in X(\lambda) \subset \mathcal{G}$ is identical to the algorithm for computing the valuations of the d_i in $\iota_{\gamma_i}(g_i)$ corresponding to $u \in P(\mathfrak{o}_S) \setminus G(\mathfrak{o}_S)$ in the bijection of Corollary 5.8. Thus the two sets of integers agree and, according to Theorem 8.2, the valuations of the d_i for a fixed prime in \mathfrak{o}_S are the \mathbf{i} -Lusztig data.

Given $\mathbf{i} = i_1 \cdots i_N$, define the ordered set of positive roots as in Section 5:

$$\gamma_1 = \alpha_{i_1}, \quad \gamma_2 = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \gamma_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}). \quad (46)$$

To compute the one-skeleton length data n_\bullet for $u \in X(\lambda) \subset U \subset \mathcal{G}$, we must first determine the values of λ_w in (45) for the GGMS stratum. Write

$$u = e_{\gamma_N}(x_N) \cdots e_{\gamma_1}(x_1), \quad x_j \in \mathbb{C}((t)). \quad (47)$$

By definition, $\lambda_{s_{i_N}}$ is the exponent λ in the torus component of the equality

$$u = t^\lambda s_{i_N} u' s_{i_N}^{-1},$$

as points in \mathcal{G} . Thus λ is easily extracted using the Iwasawa decomposition on $s_{i_N}^{-1} u s_{i_N}$. But using (47),

$$s_{i_N}^{-1} u s_{i_N} = e_{-\gamma_N}(x_N) s_{i_N}^{-1} e_{\gamma_{N-1}}(x_{N-1}) \cdots e_{\gamma_1}(x_1) s_{i_N} = e_{-\alpha_{i_N}}(x_N) u'', \quad (48)$$

for some element $u'' \in U$. Here we have used the fact that $\gamma_N = \alpha_{i_N}$, which can be seen from applying w_0 to γ_N in the form (46). Thus we reduce to a rank one Iwasawa decomposition for $e_{-\gamma_N}(x_N)$. Write $x_N = y_N/z_N$ with $y_N, z_N \in \mathfrak{o} = \mathbb{C}[[t]]$ and coprime and find elements a_N and b_N in \mathfrak{o} so that $a_N z_N + b_N y_N = 1$. Then the resulting Iwasawa decomposition (for the embedded version of this rank one decomposition) is:

$$\begin{pmatrix} 1 & \\ x_N & 1 \end{pmatrix} = \begin{pmatrix} z_N & -b_N \\ y_N & a_N \end{pmatrix} \begin{pmatrix} z_N^{-1} & \\ & z_N \end{pmatrix} \begin{pmatrix} 1 & z_N b_N \\ & 1 \end{pmatrix}. \quad (49)$$

The rightmost two matrices were called $h(z_N^{-1})e(z_N b_N)$ in Section 5. Thus the length of the one-skeleton segment connecting $\lambda_{s_{i_N}}$ and $\lambda_e = 0$ is just $\text{ord}_t(z_N)$, according to (44) with $w = e$.

To find $\lambda_{s_{i_N}s_{i_{N-1}}}$, we further conjugate by $s_{i_{N-1}}$ in (48), ignoring the embedded matrix from the right-hand side of (49) in $G(\mathbb{C}[[t]])$, which is invariant under conjugation by the Weyl group, and moving the one parameter subgroup $e_{\gamma_{N-1}}(x_{N-1})$ in the Borel leftward. Thus as a first simple step,

$$s_{i_{N-1}}h_{\gamma_N}(z_N^{-1})e_{\gamma_N}(z_N b_N)s_{i_N}e_{\gamma_{N-1}}(x_{N-1}) \cdots e_{\gamma_1}(x_1)s_{i_N}s_{i_{N-1}} = s_{i_{N-1}}s_{i_N}h_{\gamma_N}(z_N)e_{-\gamma_N}(z_N b_N)e_{\gamma_{N-1}}(x_{N-1}) \cdots e_{\gamma_1}(x_1)s_{i_N}s_{i_{N-1}}. \quad (50)$$

Then moving the h_{γ_N} and $e_{-\gamma_N}$ rightward changes the arguments of the e_{γ_i} . Indeed, the right-hand side of (50) can be rewritten using the Steinberg relations (17) and (20) and some judicious regrouping as:

$$\begin{aligned} & [s_{i_{N-1}}s_{i_N}e_{\gamma_{N-1}}(x'_{N-1})s_{i_N}s_{i_{N-1}}][s_{i_{N-1}}s_{i_N}h_{\gamma_N}(z_N)e_{-\gamma_N}(z_N b_N)s_{i_N}s_{i_{N-1}}] \\ & [(s_{i_N}s_{i_{N-1}})^{-1}e_{\gamma_{N-2}}(x'_{N-2}) \cdots e_{\gamma_1}(x'_1)s_{i_N}s_{i_{N-1}}] = \\ & e_{-\alpha_{i_{N-1}}}(x'_{N-1})[s_{i_{N-1}}s_{i_N}h_{\gamma_N}(z_N)e_{-\gamma_N}(z_N b_N)s_{i_N}s_{i_{N-1}}]u''', \quad (51) \end{aligned}$$

upon simplifying some of the bracketed terms on the left-hand side. Here x'_j with $j \leq N-1$ are elements of $\mathbb{C}((t))$ and u''' is an element of U . In the above, Lemma 5.1 has been invoked to guarantee that the relations produce the above form. Then $s_{i_N}s_{i_{N-1}}\lambda_{s_{i_N}s_{i_{N-1}}}(s_{i_N}s_{i_{N-1}})^{-1}$ may be obtained by performing another rank one Iwasawa decomposition on $e_{-\alpha_{i_{N-1}}}(x'_{N-1})$ and extracting the torus component from the element in brackets on the right-hand side of (51). Indeed, the term in brackets (upon conjugating by $s_{i_N}s_{i_{N-1}}$) gives the inductively determined $\lambda_{s_{i_1}}$ and so the length $n_{s_{i_N}s_{i_{N-1}}}$ of the edge in the one-skeleton given by

$$\lambda_{s_{i_N}s_{i_{N-1}}} - \lambda_{s_{i_N}} = n_{s_{i_N}s_{i_{N-1}}}s_{i_N}(\alpha_{i_{N-1}}^\vee)$$

is obtained solely from the torus component in the Iwasawa decomposition of $e_{-\alpha_{i_{N-1}}}(x'_{N-1})$. Using the above notation, this is just $\text{ord}_t(z'_{N-1})$.

Continuing in this manner, the terms λ_w in λ_{\bullet}^i may be inductively determined. The properties of the convex ordering from i in Lemma 5.1 again guarantee that the algorithm of pushing $e_{-\gamma_j}$'s past e_{γ_k} 's terminates in finitely many steps.

We now compare these one-skeleton lengths n_{\bullet} to the parametrization of elements of the flag variety in Section 5. Recall that we found representatives for elements in the big cell of the flag variety $P(\mathfrak{o}) \backslash G(\mathfrak{o}) \cap P(F)U_-^P(F)$ as a product of elements in embedded $SL_2(\mathfrak{o})$'s according to the reduced word i . More precisely, Corollary 5.8 established a bijection between bottom rows (c_j, d_j) with $j \in [1, N]$ of the embedded $SL_2(\mathfrak{o})$ matrices with points in the big cell of the flag variety. These bottom row elements (c_i, d_i) are in bijection with elements of $U_-^P(F)$ by Corollary 5.3. In Theorem 5.2, we explained how to determine the pairs (c_j, d_j) corresponding to an element $u \in U_-^P(F)$. The process of determining d_j in the proof of Theorem 5.2 is formally identical to the process of determining the ring element z_j of $\mathbb{C}[[t]]$ appearing in the Iwasawa decomposition of $e_{-\alpha_{i_j}}$ at each stage above. As we are only interested in the valuation of the d_i at a fixed prime p in \mathfrak{o}_S , we may pass to the localization and formally identify the set of possible valuations of the resulting uniformizers p and t . Thus we arrive at the following result.

Theorem 8.3. *Given a reduced word \mathbf{i} , let (c_j, d_j) for $j = 1, \dots, N$ be the corresponding coordinates for points in the big cell of the flag variety as in Corollary 5.8. Then for a fixed prime $p \in \mathfrak{o}$, $n_\bullet = (\text{ord}_p(d_1), \dots, \text{ord}_p(d_N))$ is the \mathbf{i} -Lusztig data. In particular, if \mathbf{i}' is any other reduced word, with coordinates (c'_j, d'_j) with valuations of the d'_j written as n'_\bullet , then n_\bullet and n'_\bullet are related by the Lusztig transition maps $R_{\mathbf{i}}^{\mathbf{i}'}$ as in (42), (43), and Section 3 of [4].*

8.3. String data for canonical bases. While the moduli of the exponential sum $S_{\ell, m}$ are given by Lusztig data, the evaluation of the sum is more easily described in terms of an alternate basis for the module \mathcal{B}^\vee of $(\mathcal{U}^\vee)^+$ – the so-called “string bases” of Kashiwara. Their definition is recalled now. We illustrate each general observation with a running example in GL_4 ; the conclusion of an example is marked by Δ .

Let b be any vector in a canonical basis \mathcal{B}_λ^\vee for a finite dimensional representation of the dual group G^\vee with highest weight λ . Explicitly, \mathcal{B}_λ^\vee is the set of all $b \in \mathcal{B}^\vee$ such that $b \cdot v_\lambda \neq 0$, where v_λ is the highest weight vector in the module. Given a word $\mathbf{i} = (i_1, \dots, i_\nu)$ corresponding to a reduced decomposition of the long element w_0 , let n_1 be the maximal integer such that

$$b_1 := F_{i_1}^{(n_1)}(b) \neq 0$$

where F_i denotes the Kashiwara operator for the simple root α_i . Now let n_2 be the maximal integer such that $F_{i_2}^{(n_2)}(b_1) \neq 0$. Repeating for each index, we obtain a sequence of integers (n_1, \dots, n_ν) and the resulting vector b_ν is the lowest weight vector in the representation. The integers (n_1, \dots, n_ν) for each vertex b in \mathcal{B}_λ^\vee will be referred to as “string data.” Let \mathcal{C}_i^λ be the set of all such ν -tuples of integers over all elements of \mathcal{B}_λ^\vee .

The following is a special case of a result of Berenstein-Zelevinsky and Littelmann may be found, for example, in Corollary 1 of Theorem 1.7 in Littelmann [27].

Theorem 8.4 (Berenstein-Zelevinsky, Littelmann). *The integer lattice points \mathcal{C}_i^λ as defined above determine a convex polytope in \mathbb{R}^N .*

Thus we refer to the resulting polytope as the “BZL polytope” corresponding to reduced word \mathbf{i} and dominant weight λ . Moreover, Littelmann showed (see Theorem 4.2 of [27]) that if \mathbf{i} is chosen with respect to a sequence of braidless maximal parabolics (see Definition 5.4), then the corresponding inequalities defining the polytope have a particularly simple form. More precisely, let $B \subset P_1 \subset P_2 \subset \dots \subset G^\vee$ be a sequence of braidless maximal parabolics. Here we mean that P_i is maximal in P_{i+1} . Let W_i be the Weyl group generated by simple reflections in P_i and iW_i the set of minimal length coset representatives in $W_{i-1} \backslash W_i$. Then we choose a reduced expression for

$$w_0 = \underbrace{s_{i_1}}_{\tau_1} \underbrace{s_{i_2} \cdots}_{\tau_2} \cdots \underbrace{s_{i_j} \cdots s_{i_\nu}}_{\tau_r} \quad (52)$$

such that τ_i is the longest word in iW_i .

Example: For GL_4 , consider the strictly dominant highest weight $\lambda + \rho$ where

$$\lambda = m_1 \epsilon_1 + m_2 \epsilon_2 + m_3 \epsilon_3, \quad \rho = \epsilon_1 + \epsilon_2 + \epsilon_3, \quad \epsilon_i : \text{fundamental weights, } m_i \geq 0.$$

The reduced word $\mathbf{i} = (1, 3, 2, 1, 3, 2)$ corresponds to such a sequence of braidless maximal parabolics as in (52). The polytope inequalities for the string data (n_1, \dots, n_6) also respect the sequence of parabolics. Focusing on the string data (n_3, n_4, n_5, n_6) corresponding to the

maximal parabolic P in G^\vee with $\mathbf{i}^P = \{2, 1, 3, 2\}$, we describe the resulting 4-dimensional polytope.

By Theorem 7.1 of [27], the cone inequalities satisfied by all string data in the canonical basis \mathcal{B}^\vee are

$$n_3 \geq \max(n_4, n_5), \quad \min(n_4, n_5) \geq n_6 \geq 0 \quad (53)$$

Then imposing the highest weight $\lambda + \rho$ as above, according to Section 7, Corollary 1 of [27], the string data for the finite crystal $\mathcal{B}_{\lambda+\rho}^\vee$ satisfies the highest weight inequalities:

$$n_6 \leq m_2 + 1 \quad (54)$$

$$n_5 \leq m_3 + 1 + n_6 \quad (55)$$

$$n_4 \leq m_1 + 1 + n_6 \quad (56)$$

$$n_3 \leq m_2 + 1 + n_4 + n_5 - 2n_6. \quad (57)$$

The indexing above differs from [27] because our string data is recorded with lowering operators, while [27] uses raising operators. \triangle

Returning to the general case, it remains to describe the algorithm for mapping from \mathbf{i} -Lusztig data to string data. To do so, we must describe how to apply the Kashiwara lowering operators F_{i_j} to the basis element $E_{\mathbf{i}}^c$ in Lusztig's basis. According to the form of (41), it is easy to determine the effect of F_{i_1} on $E_{\mathbf{i}}^c$. The integer c_1 is maximal such that

$$F_{i_1}^{(c_1)} E_{i_1}^{(c_1)} T_{i_1}(E_{i_2}^{(c_2)}) \cdots (T_{i_1} T_{i_2} \cdots T_{i_{\nu-1}})(E_{i_\nu}^{(c_\nu)}) \neq 0.$$

After applying F_{i_1} to $E_{\mathbf{i}}^c$ c_1 times, the resulting \mathbf{i} -Lusztig data is

$$(0, c_2, \dots, c_\nu)$$

In order to determine the effect of F_{i_2} on the resulting basis element, we use a sequence of braid relations to move from Lusztig data for the long word $\mathbf{i} = (i_1, i_2, \dots)$ to Lusztig data for $\mathbf{i}' = (i_2, \dots)$. This can be done since any long word is related to any other by a sequence of braid relations and there exists a long word beginning with any given simple reflection. For example, the transition maps for the case of G^\vee simply laced were given in (42) and (43). Repeating this for each of the successive i_j appearing in \mathbf{i} , then the algorithm terminates at the lowest weight vector and returns string data for the corresponding basis vector $E_{\mathbf{i}}^c$.

Example: $G = GL_4$ with long word $\mathbf{i} = (1, 3, 2, 1, 3, 2)$.

Beginning with an element b with Lusztig data (c_1, \dots, c_6) , then applying the F_1 operator c_1 times, we arrive at a basis vector b_1 with \mathbf{i} -Lusztig data $(0, c_2, \dots, c_6)$. Then since the simple reflections s_1 and s_3 commute, the transition map (42) is of the form

$$R_{\mathbf{i}}^{\mathbf{i}'} : (0, c_2, c_3, \dots, c_6) \mapsto (c_2, 0, c_3, \dots, c_6)$$

where $\mathbf{i}' = (3, 1, 2, 1, 3, 2)$. Then F_3 may be applied c_2 times to the above vector resulting in a basis vector b_2 with Lusztig data $(0, 0, c_3, \dots, c_6)$. Now one must move from \mathbf{i}' to a word beginning with 2 to apply the F_2 operator. This may be accomplished by the sequence of moves:

$$\mathbf{i}' \mapsto (3, 2, 1, 2, 3, 2) \mapsto (3, 2, 1, 3, 2, 3) \mapsto (3, 2, 3, 1, 2, 3) \mapsto (2, 3, 2, 1, 2, 3).$$

Keeping track of the effect on the Lusztig data, the first application of the braid relation maps

$$(0, 0, c_3, c_4, c_5, c_6) \mapsto (0, c_3 + c_4, 0, c_3, c_5, c_6)$$

according to (43). Continuing in this manner, the resulting string data takes the form

$$(n_1, n_2, n_3, n_4, n_5, n_6) = (c_1, c_2, c_4 + c_5 + \max(c_3, c_6), c_3 + c_5, c_3 + c_4, \min(c_3, c_6)). \quad (58)$$

△

In general, this map from the Lusztig data to string data respects parabolic induction. That is, let $w_0 = w_M w^P$ be a factorization of the long element corresponding to a Levi decomposition of the parabolic $P = MU^P$. Let \mathbf{i} be a reduced word for w_0 , chosen so that

$$\mathbf{i} = \mathbf{i}_M \cdot \mathbf{i}^P, \quad \text{where “} \cdot \text{” denotes concatenation,}$$

and \mathbf{i}_M is a long word for M . Then under the map described above, the string data for \mathbf{i}^P will only involve Lusztig data for the long word \mathbf{i}^P . Indeed, M is a reductive group and so the algorithm, which is performed first for \mathbf{i}_M , invokes only braid relations within \mathbf{i}_M . After completing the algorithm for each of the simple reflections in w_M , then we arrive at a basis element with Lusztig data beginning with a list of 0's at each position corresponding to i_j in \mathbf{i}_M and leaving the original Lusztig data with index in \mathbf{i}^P unchanged.

Example: $G = GL_4$ and $M = GL_2 \times GL_2$ then the word

$$(1, 3, 2, 1, 3, 2) = (1, 3) \cdot (2, 1, 3, 2)$$

is indeed compatible with the choice of parabolic with Levi factor M . Then with respect to $\mathbf{i}^P = (2, 1, 3, 2)$, the map given in (58) shows that the string data (n_3, \dots, n_6) can indeed be given using only the Lusztig data (c_3, \dots, c_6) .

Returning to Whittaker coefficients associated to this example, note that the ℓ in the exponential sum $S_{\ell, \mathbf{m}}$ of (38) may be considered as attached to Lusztig data \mathbf{c} (according to our numbering scheme) as follows:

$$\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, \ell_1, \ell_2, \ell_3, \ell_4).$$

And so this may, in turn, be mapped to string data (relative to the parabolic P) via (58) with

$$(n_3, n_4, n_5, n_6) = (\ell_2 + \ell_3 + \max(\ell_1, \ell_4), \ell_1 + \ell_3, \ell_1 + \ell_2, \min(\ell_1, \ell_4)) \quad (59)$$

Identifying \mathbf{m} with the weight $\lambda = m_1 \epsilon_1 + m_2 \epsilon_2 + m_3 \epsilon_3$, we write $\mathcal{B}_{\mathbf{m}+\rho}^\vee$ for the canonical basis elements in the corresponding highest weight module. The inequalities (54)–(57) for string data in $\mathcal{B}_{\mathbf{m}+\rho}^\vee$ may now be rewritten in terms of Lusztig data using (59) and a simple case analysis according to the value of $\min(\ell_1, \ell_4)$. Then the highest-weight inequalities are equivalent to:

$$\ell_4 \leq m_2 + 1 \quad (60)$$

$$\ell_1 + \ell_2 \leq m_3 + 1 + \min(\ell_1, \ell_4) \quad (61)$$

$$\ell_1 + \ell_3 \leq m_1 + 1 + \min(\ell_1, \ell_4). \quad (62)$$

Thus, the above three inequalities (together with the obvious requirement that $\ell_i \geq 0$) cut out the finite set of \mathbf{i}^P -Lusztig data for the elements of $\mathcal{B}_{\mathbf{m}+\rho}^\vee$. In Section 10 below, we use these inequalities to describe the support of the exponential sum $S_{\ell, \mathbf{m}}$. △

9. GENERIC EVALUATION OF THE EXPONENTIAL SUM AT PRIME POWERS

In this section, given a reductive group G , any cominuscle parabolic subgroup P , and a fixed prime $p \in \mathfrak{o}_S$, we evaluate the sum $S_{\ell, \mathbf{m}}$ defined in (35) for many choices of ℓ and \mathbf{m} . Recall that $\ell = (\ell_1, \dots, \ell_N)$ with $\ell_i = \text{ord}_p(d_i)$ and $\mathbf{m} = (m_1, \dots, m_r)$ with $m_j = \text{ord}_p(t_j)$. We establish two results. The first states that, for certain hyperplane inequalities on ℓ and \mathbf{m} , the exponential sum $S_{\ell, \mathbf{m}}$ has a very regular evaluation given in terms of the Euler phi-function. The second states that for the reverse inequality, now with respect to ℓ and $\mathbf{m} + \rho = (m_1 + 1, \dots, m_r + 1)$, the exponential sum “generically” vanishes. More precisely, it vanishes for these ℓ and \mathbf{m} outside of a set of integer lattice points lying on certain hyperplanes in r -dimensional space. These are almost immediate consequences of the formula for the exponential sum as given in (29), but we have postponed them to the present section to discuss relations between these hyperplane inequalities and canonical bases (see Remark 9.3 at the end of this section).

We begin by rewriting (29) when \mathbf{d} and \mathbf{t} are prime powers as above, and obtain the general expression

$$S_{\ell, \mathbf{m}} = \sum_{c_j \pmod{p^{\text{ord}_p(D_j)}}} \psi \left(\sum_j p^{m_j} v_j \right) \prod_{k=1}^N \left(\frac{c_k}{p^{\ell_k}} \right)^{q_k}. \quad (63)$$

Recall that

$$\text{ord}_p(D_j) = \ell_j + \sum_{i=j+1}^N \langle \gamma_j, \gamma_i^\vee \rangle \ell_i \quad \text{and} \quad v_j = \sum_{(k, k') \in \mathcal{S}_j} \left[\eta_{i, i', k, -k'} b_k c_{k'} p^{-\text{ord}_p(D(k, k'; \alpha_j))} \right] \quad (64)$$

for each root simple root $\alpha_j \in P$ according to (25), noting that by Proposition 6.4 the pairs (i, i') used in linear combinations of roots for \mathcal{S}_j are both 1 if P is cominuscle. The element $D(k, k'; \alpha_j)$ is given explicitly in Lemma 6.3. If α_i is the unique simple root omitted from P , recall that it appears as γ_N in our ordering and thus $\text{ord}_p(D_N) = \ell_N$ and $v_i = c_N p^{-\ell_N}$.

We now observe that within certain limits on the ℓ_j , the coefficients $S_{\ell, \mathbf{m}}$ are supported on an N -dimensional rectangular lattice with side lengths $n(\gamma_j)$ where we define $n(\alpha) := n / \gcd(n, Q(\alpha^\vee))$ for any root α . The evaluation of $S_{\ell, \mathbf{m}}$ at these lattice points is given in terms of the Euler phi function on $p^r \mathfrak{o}_S$ for non-negative integers r , denoted $\phi(p^r)$.

Proposition 9.1. *Suppose that for all simple roots $\alpha_j \in P$ and pairs $(k, k') \in \mathcal{S}_j$, we have $m_j \geq \text{ord}_p(D(k, k'; \alpha_j))$, and that for the omitted root α_i , $m_i \geq \ell_N$. Then*

$$S_{\ell, \mathbf{m}} = \begin{cases} \prod_{k=1}^N |p|^{\epsilon_k} \phi(p^{\ell_k}) & \text{if } n(\gamma_k) \mid \ell_k \text{ for all } k = 1, \dots, N \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\epsilon_k = \sum_{i=k+1}^N \langle \gamma_k, \gamma_i^\vee \rangle \ell_i.$$

Proof. With these assumptions on m_j , the character ψ appearing in (63) is identically 1 in $S_{\ell, \mathbf{m}}$. The result follows immediately, since the multiplicative character is trivial exactly when $n(\gamma_k) \mid \ell_k$ for all $k = 1, \dots, N$. \square

Proposition 9.2. *Suppose that at least one of the following conditions hold:*

- (1) *There exists a simple root $\alpha_j \in P$ and a pair $(k, k') \in \mathcal{S}_j$ with $\ell_k, \ell_{k'} > 0$ such that the difference*

$$\ell_{k,k';j} := \text{ord}_p(D(k, k'; \alpha_j)) - m_j$$

is greater than one. Moreover, this difference is larger than $\ell_{k_0, k'_0; j_0}$ for all simple roots $\alpha_{j_0} \in P$ and all pairs (k_0, k'_0) in \mathcal{S}_{j_0} with one of k_0, k'_0 equal to k' .

- (2) *One has $\ell_N - m_i > 1$, where α_i is the simple root omitted from P , and this difference is larger than $\ell_{k_0, k'_0; j_0}$ for all simple roots $\alpha_{j_0} \in P$ and all pairs (k_0, k'_0) in \mathcal{S}_{j_0} with one of k_0, k'_0 equal to N .*

Then $S_{\ell, \mathbf{m}} = 0$.

Proof. Suppose first that $\ell_{k,k';j} > 1$ for some j and a pair $(k, k') \in \mathcal{S}_j$ and satisfies the above maximality condition. Then we may perform the changes of variables

$$c_{k'} \mapsto c_{k'}(1 + rp^{\ell_{k,k';j}-1})$$

for each $r \bmod p$. Summing the result, we obtain $|p|$ times the original sum $S_{\ell, \mathbf{m}}$. On the other hand, according to (63), a factor of

$$\psi \left(\frac{\eta_{i,i',k,-k'} b_k c_{k'} r}{p} \right)$$

pulls out of each summand for each $r \bmod p$. This is true since, in all other occurrences of $c_{k'}$ or $b_{k'}$ in ψ , the character ψ is invariant under this substitution according to our maximality assumption and moreover, the power residue symbols are unchanged. Since ℓ_k and $\ell_{k'}$ are positive, the integers b_k and $c_{k'}$ are coprime to p and hence non-zero. Thus summing first over r , the sum $S_{\ell, \mathbf{m}}$ vanishes.

If instead $\ell_N - m_i > 1$ where α_i is the omitted simple root from P and the difference is larger than all $\ell_{k_0, k'_0; j_0}$ as above, then a similar change of variables $c_N \mapsto c_N(1 + rp^{\ell_N - m_i - 1})$ causes $S_{\ell, \mathbf{m}}$ to vanish by the same argument as in the previous case. \square

Remark 9.3. In Theorem 8.3, we explained that ℓ gives the Lusztig data for the reduced word \mathbf{i}^P . We conjecture that the inequalities $m_i \geq \ell_N$ with $\alpha_i \notin P$ and $m_j \geq \text{ord}_p(D(k, k'; \alpha_j))$ for $\alpha_j \in P$ and $(k, k') \in \mathcal{S}_j$ used in Propositions 9.1 and 9.2 are precisely the inequalities for \mathbf{i}^P -Lusztig data in the representation of G^\vee with highest weight \mathbf{m} . Surprisingly, these do not appear to have been written down explicitly in the literature. In principle, they may be extracted from [5] which gives a polytope realization of string data ([5], Theorem 3.10), proves that the string cones factor when the long word factors according to parabolic induction ([5], Theorem 3.11), and gives explicit bijections between string and Lusztig data via use of natural “twist” automorphisms ([5], Theorem 5.7). The results are stated for the canonical bases \mathcal{B}^\vee but are of course compatible with highest weight modules by taking those $b \in \mathcal{B}^\vee$ such that $b \cdot v_{\text{high}} \neq 0$, where v_{high} is the highest weight vector of the module. However, making these maps explicit in terms of the root datum would take us far afield, and we shall not carry it out here.

Morier-Genoud [37] has applied the ideas of [5] to study generalizations of the Schützenberger involution for highest weight modules of semisimple groups. Though not addressing the above question, many of the quantities obtained are similar to those of Section 5. For example, our formula for $\text{ord}_p(D_j)$ given in (64) agrees with (2.12) in [37].

Finally, we note that the relation between highest weight inequalities for Lusztig data and the hyperplanes $m_j \geq \text{ord}_p(D(k, k'; \alpha_j))$ can be checked directly in our running GL_4 example, using (60), (61), and (62). Moreover, the results reviewed in Section 8.3 provide an algorithm for directly checking this identity for any group G and parabolic P .

In summary, we have evaluated the exponential sum $S_{\ell, \mathbf{m}}$ for fixed \mathbf{m} except for those ℓ such that either $\text{ord}_p(D(k, k'; \alpha_j)) = m_j + 1$ for some $\alpha_j \in P$ and $(k, k') \in \mathcal{S}_j$, or $\ell_N = m_i + 1$ with $\alpha_i \notin P$, or else ℓ lies on one of the exceptional hyperplanes

$$\ell_{k, k'; j} = \ell_{k_0, k'_0; j_0} \quad \alpha_j, \alpha_{j_0} \in P; (k, k'), (k_0, k'_0) \in \mathcal{S}_{j_0}; k_0 = k' \text{ or } k'_0 = k'$$

and

$$\ell_N - m_i = \ell_{k_0, k'_0; j_0} \quad \alpha_i \notin P, \alpha_{j_0} \in P; (k_0, k'_0) \in \mathcal{S}_{j_0}; k_0 = k' \text{ or } k'_0 = N,$$

which moreover satisfy the maximality conditions in Proposition 9.2. To give a flavor of the complexity in these remaining cases, we carry out a full analysis in the GL_4 example.

10. THEOREMS ON THE SUPPORT OF \widetilde{GL}_4 EXPONENTIAL SUMS

The goal of the next two subsections is to prove the following theorem, which demonstrates that the exponential sum $S_{\ell, \mathbf{m}}$ in our recurring \widetilde{GL}_4 example is supported on the union of the i_P -Lusztig data for the elements of the finite crystal $\mathcal{B}_{\mathbf{m}+\rho}^\vee$ and an infinite collection of integer lattice points lying on a particular 2-dimensional hyperplane.

Theorem 10.1. *The sum $S_{\ell, \mathbf{m}}$ defined in (38) is zero unless either*

- a) the highest weight inequalities (60), (61), (62) hold, or*
- b) $\ell_1 = \ell_4 \leq m_2 + 1$ and $\ell_2 - m_3 = \ell_3 - m_1 > 1$.*

In the course of proving this theorem, we also provide an explicit evaluation of $S_{\ell, \mathbf{m}}$ in case (b). In Section 10.3, we use this explicit formula to explain why the contribution from case (b) does not contribute to the Whittaker function. The evaluation of the sum in case (a) is postponed until Section 11. In both this section and the next, the evaluations will be defined in terms of n -th order Gauss sums. For non-negative integers m, ℓ , an integer $j \bmod n$, and additive character ψ on F_S of conductor \mathfrak{o}_S as before, we use the notation

$$g_j(m, \ell) := \sum_{\substack{a \bmod p^\ell \\ (a, p)=1}} \left(\frac{a}{p}\right)^{j\ell} \psi\left(\frac{p^m a}{p^\ell}\right),$$

using the n -th power residue symbol as in Section 2.2. Here we assume both the degree of the cover n and the prime $p \in \mathfrak{o}_S$ to be fixed throughout. As a further shorthand, let

$$g(k) := g_1(k-1, k), \quad h(k) := g_1(k, k), \quad g(m, \ell) := g_1(m, \ell). \quad (65)$$

This choice of notation is reasonable since our answers will be given uniformly for all n and all $p \in \mathfrak{o}_S$ in terms of such Gauss sums, though of course their explicit evaluation as complex numbers depends on this data. For example, $h(k)$ is a degenerate Gauss sum, equal to the Euler phi function $\phi(p^k)$ of $p^k \mathfrak{o}_S$ if n divides k and to 0 otherwise. By contrast, $g(k)$ does not admit a simpler explicit description for $n > 2$ unless $n \mid k$.

10.1. Proof of Theorem 10.1: The Case of Positive Lusztig Data. In this section, we prove Theorem 10.1 in the case that all $\ell_i > 0$. This condition guarantees that each of the c_i and b_i are relatively prime to p . The exponential character is the product of three terms:

$$\psi \left(-p^{m_1} \left(\frac{b_2 c_1 p^{\ell_4}}{p^{\ell_1 + \ell_3}} + \frac{b_4 c_3}{p^{\ell_3}} \right) \right) \cdot \psi \left(p^{m_2} \frac{c_4}{p^{\ell_4}} \right) \cdot \psi \left(p^{m_3} \left(\frac{c_1 b_3 p^{\ell_4}}{p^{\ell_1 + \ell_2}} + \frac{c_2 b_4}{p^{\ell_2}} \right) \right).$$

Consider the following change in the c_2 variable:

$$c_2 \mapsto c_2(1 + ap^{\max(\ell_2 - m_3, 1)}), \quad \text{for any } a \bmod p.$$

The third factor doesn't change, and the power residue symbol is also unchanged. Hence analyzing the first term in the exponential, we obtain the following inequality as a necessary condition for the sum to be non-zero:

$$\ell_1 + \ell_3 \leq m_1 + \ell_4 + \max(\ell_2 - m_3, 1). \quad (66)$$

To analyze the inequality (66) further, we separate the following two cases for the value of $\max(\ell_2 - m_3, 1)$.

$$\textbf{CASE A1: } \ell_2 \leq m_3 + 1. \quad \textbf{CASE A2: } \ell_2 > m_3 + 1.$$

In Case A1, (66) gives $\ell_1 + \ell_3 \leq m_1 + \ell_4 + 1$. Note that in Case A1, (61) is satisfied if $\min(\ell_1, \ell_4) = \ell_1$, and (62) is satisfied if $\min(\ell_1, \ell_4) = \ell_4$.

In Case A2, let us make the change of variables

$$c_2 \mapsto c_2(1 + ap^{\ell_2 - m_3 - 1}), \quad \text{for any } a \bmod p.$$

Then b_2 gets replaced by $b_2(1 - ap^{\ell_2 - m_3 - 1} + O(p^{2(\ell_2 - m_3 - 1)}))$. Then the dependence on a is given by

$$\psi \left(\frac{ab_4 c_2}{p} + \frac{ac_1 b_2 p^{m_1 + \ell_4 + \ell_2 - m_3 - 1}}{p^{\ell_1 + \ell_3}} \right)$$

times a term involving a larger power of p that will not change the subsequent analysis. As we sum over a , such an exponential gives 0 unless both displayed terms have the same power of p in the denominator. This implies that the support of the sum in Case A2 satisfies $\ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3$.

The variables c_2 and c_3 have symmetric roles in the exponential sum. Thus we may perform analogous substitutions for c_3 and obtain the pair of conditions and their consequences when they are in the support of $S_{\ell, m}$:

$$\textbf{CASE B1: } \ell_3 \leq m_1 + 1 \implies \ell_1 + \ell_2 \leq m_3 + \ell_4 + 1$$

$$\textbf{CASE B2: } \ell_3 > m_1 + 1 \implies \ell_1 + \ell_2 = m_3 + \ell_4 + \ell_3 - m_1.$$

Symmetrically to the above, in Case B1, (62) is satisfied if $\min(\ell_1, \ell_4) = \ell_1$, and (61) is satisfied if $\min(\ell_1, \ell_4) = \ell_4$.

We now analyze various combinations of Cases A and B.

Lemma 10.2. *Suppose that $S_{\ell, m} \neq 0$. If the inequality A1 holds, then B1 must hold. Similarly, if A2 holds, then B2 must hold.*

Proof. The two cases are symmetric; we do the second. Suppose that A2 and B1 hold and $S_{\ell, \mathbf{m}} \neq 0$. Then we have the conditions

$$\begin{aligned} \ell_2 &> m_3 + 1 \quad \text{and} \quad \ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3 \\ \ell_3 &\leq m_1 + 1 \quad \text{and} \quad \ell_1 + \ell_2 \leq m_3 + \ell_4 + 1. \end{aligned}$$

The conditions $\ell_2 > m_3 + 1$ and $\ell_1 + \ell_2 \leq m_3 + \ell_4 + 1$ imply that $\ell_1 < \ell_4$. But $\ell_3 \leq m_1 + 1$, so this implies that $\ell_1 + \ell_3 < \ell_4 + m_1 + 1$. Combining this with $\ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3$ gives $\ell_2 \leq m_3$, a contradiction. \square

Lemma 10.3. *If the A1 and B1 inequalities hold and the sum $S_{\ell, \mathbf{m}} \neq 0$, then the highest weight inequalities (60), (61), (62) hold.*

Proof. The dependence of the character ψ on b_4, c_4 is given by

$$\psi \left(p^{m_2 - \ell_4} c_4 + b_4 \left(-c_3 p^{m_1 - \ell_3} + c_2 p^{m_3 - \ell_2} \right) \right).$$

The hypotheses guarantee that $\text{ord}_p(-c_3 p^{m_1 - \ell_3} + c_2 p^{m_3 - \ell_2}) \geq -1$. But the sum

$$\sum_{c_4 \bmod p^{\ell_4}, (c_4, p)=1} \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(\frac{c_4}{p^{2+r}} + \frac{b_4}{p^{1-s}} \right)$$

is zero for any $r, s \geq 0$, as the variable change $c_4 \mapsto c_4(1 + ap^{1+r})$ readily shows. Hence we must have $\ell_4 \leq m_2 + 1$ in order for the sum over c_4 to be non-vanishing. Under the A1, B1 inequalities, (61) and (62) also hold, since no matter which of ℓ_1, ℓ_4 is their minimum, one of these inequalities follows from A1 and the other from B1. \square

Lemma 10.4. *If the A2 and B2 inequalities hold, then the exponential sum $S_{\ell, \mathbf{m}}$ vanishes unless*

$$\ell_1 = \ell_4, \quad m_1 + \ell_2 = m_3 + \ell_3. \quad (67)$$

In that case, using the notation in (65),

$$S_{\ell, \mathbf{m}} = q^{\ell_2 + \ell_3 + 2\ell_4 - k - \ell_1} g_0(m_2, \ell_4) h(2\ell_1 + \ell_2 + \ell_3) \quad (68)$$

with $k = \ell_2 - m_3 = \ell_3 - m_1$ (and $g_0(m_2, 0) = 1$ by definition).

Proof. We consider the possible support of $S_{\ell, \mathbf{m}}$ in this case. Since $\ell_1 + \ell_3 = m_1 + \ell_4 + \ell_2 - m_3$ and $\ell_1 + \ell_2 = m_3 + \ell_4 + \ell_3 - m_1$, the equalities (67) follow at once. Then the dependence of the summand on c_1 is given by

$$\left(\frac{c_1}{p^{\ell_1}} \right) \psi \left(-p^{m_1} \left(\frac{b_2 c_1 p^{\ell_4}}{p^{\ell_1 + \ell_3}} \right) + p^{m_3} \left(\frac{c_1 b_3 p^{\ell_4}}{p^{\ell_1 + \ell_2}} \right) \right) = \left(\frac{c_1}{p^{\ell_1}} \right) \psi(p^{-k}(b_2 - b_3)c_1),$$

with $k > 1$ as defined in the statement of the lemma. Thus the sum over c_1 will be 0 unless $b_2 \equiv b_3 \bmod p^{k-1}$. Note that

$$\sum_{c_1} \left(\frac{c_1}{p^{\ell_1}} \right) \psi(p^{-k}(b_2 - b_3)c_1)$$

is equal to

$$q^{\ell_2 + \ell_3} \left(\frac{u^{-1}}{p^{\ell_1}} \right) g(\ell_1)$$

if $b_2 - b_3 = up^{k-1}$ with $(u, p) = 1$, and instead gives $q^{\ell_2+\ell_3}h(\ell_1)$ if $p^k \mid b_2 - b_3$. Thus the total remaining sum is:

$$q^{\ell_2+\ell_3+2\ell_4}g(\ell_1) \sum_{\substack{c_2, c_3, c_4 \\ b_2-b_3=up^{k-1} \\ (u,p)=1}} \left(\frac{u^{-1}}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(\frac{b_4(c_2 - c_3)}{p^k} + \frac{c_4 p^{m_2}}{p^{\ell_4}} \right) \\ + q^{\ell_2+\ell_3+2\ell_4}h(\ell_1) \sum_{\substack{c_2, c_3, c_4 \\ p^k \mid b_2-b_3}} \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(\frac{c_4 p^{m_2}}{p^{\ell_4}} \right).$$

We note that since $k > 1$, in both sums $b_2 \equiv b_3 \pmod p$ and so $(c_3/p^{\ell_3}) = (c_2/p^{\ell_3})$. Thus the second term gives

$$q^{\ell_2+\ell_3+2\ell_4-k}h(\ell_1)g(m_2, \ell_4)h(\ell_2 + \ell_3). \quad (69)$$

For the first term, note that if $b_2 - b_3 = up^{k-1}$ then $c_2 - c_3 = b_2^{-1} - b_3^{-1} = -c_2 c_3 up^{k-1}$ modulo p^k . Regard the sum as over c_4 , c_2 and u , and make a change of variables $u \mapsto ub_2^2 c_4$. Then the first term gives a contribution of

$$q^{\ell_2+\ell_3+2\ell_4}g(\ell_1) \sum_{\substack{c_2, c_4, u \\ u \bmod p^{\ell_3-k+1} \\ (u,p)=1}} \left(\frac{u^{-1}c_2^2 c_4^{-1}}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2+\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(\frac{-u}{p} + \frac{c_4 p^{m_2}}{p^{\ell_4}} \right). \quad (70)$$

The three sums now separate and they evaluate as follows. The c_2 sum gives $q^{-2\ell_1-\ell_3}h(2\ell_1 + \ell_2 + \ell_3)$. The sum over u gives $q^{\ell_3-k+1-\ell_1}$ times the conjugate of $g(\ell_1)$. Since $\ell_1 = \ell_4$, the multiplicative characters in c_4 cancel, and then the c_4 sum gives $g_0(m_2, \ell_4)$. Thus this term contributes

$$q^{\ell_2+\ell_3+2\ell_4-k-\ell_1}h(2\ell_1 + \ell_2 + \ell_3)g_0(m_2, \ell_4)$$

if n does not divide ℓ_1 and

$$q^{\ell_2+\ell_3+2\ell_4-k-\ell_1-1}h(2\ell_1 + \ell_2 + \ell_3)g_0(m_2, \ell_4)$$

if $n \mid \ell_1$. Adding (69) to the expressions above gives the result. (Note that when $n \mid \ell_1 = \ell_4$, one has $g(m_2, \ell_4) = g_0(m_2, \ell_4)$.) \square

This concludes the proof of Theorem 10.1 in the case when all the Lusztig data ℓ_i are positive.

10.2. Proof of Theorem 10.1: Remaining Cases. We turn to a consideration of Theorem 10.1 when at least one of the $\ell_i = 0$.

Suppose $\ell_2 = 0$. Then we may take $b_2 = c_2 = 0$ and remove (c_2/p^{ℓ_2}) . The sum becomes

$$q^{2\ell_4} \sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(-p^{m_1} \frac{b_4 c_3}{p^{\ell_3}} + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_1 b_3 p^{\ell_4}}{p^{\ell_1}} \right).$$

If in addition $\ell_3 = 0$, we may take $b_3 = c_3 = 0$ and remove (c_3/p^{ℓ_3}) . The divisibility conditions (39), (40) become $\ell_1 \leq \min(m_1, m_3) + \ell_4$. If $\ell_4 = 0$ we are done. If $\ell_4 \neq 0$, then we obtain a sum over c_4 which vanishes unless $\ell_4 \leq m_2 + 1$. So we obtain all highest weight inequalities for $\ell_2, \ell_3 = 0$.

If $\ell_2 = 0$ but $\ell_3 \neq 0$ then $(b_3, p) = 1$. The sum over c_1 vanishes unless $\ell_1 \leq m_3 + \ell_4 + 1$. If $\ell_4 = 0$ then we also have the divisibility condition $\ell_1 + \ell_3 \leq m_1$. These give the highest weight inequalities.

If $\ell_2 = 0$, $\ell_3, \ell_4 \neq 0$, we make the changes $c_1 \mapsto c_1 c_3$ and then $c_3 \mapsto c_3 c_4$. Then the sum factors. The c_1 sum gives $q^{\ell_3} g(m_3 + \ell_4, \ell_1)$, the c_3 sum gives $q^{-\ell_1} g(\ell_1 + m_1, \ell_1 + \ell_3)$, and the c_4 sum gives $q^{-\ell_1 - \ell_3} g(\ell_1 + \ell_3 + m_2, \ell_1 + \ell_3 + \ell_4)$. These sums vanish unless $\ell_3 \leq m_1 + 1$ and $\ell_4 \leq m_2 + 1$. Once again, the desired inequalities follow after incorporating the divisibility conditions. This completes all cases with $\ell_2 = 0$. The cases with $\ell_3 = 0$ are symmetric.

Suppose now $\ell_4 = 0$ but $\ell_2, \ell_3 \neq 0$. Since $\ell_4 = 0$, we may take $b_4 = c_4 = 0$ and the sum $S_{\ell, m}$ simplifies to:

$$q^{2\ell_4} \sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \psi \left(-p^{m_1} \left(\frac{b_2 c_1}{p^{\ell_1 + \ell_3}} \right) + p^{m_3} \left(\frac{c_1 b_3}{p^{\ell_1 + \ell_2}} \right) \right).$$

If $c_1 \not\equiv 0 \pmod{p}$, then we may sum over b_2, b_3 which are relatively prime to p . This sum is zero unless the following inequalities hold:

$$\ell_1 + \ell_3 \leq m_1 + 1, \quad \ell_1 + \ell_2 \leq m_3 + 1. \quad (71)$$

These imply the desired highest weight inequalities. This condition on c_1 is guaranteed whenever $\ell_1 > 0$. Thus we have established Theorem 10.1 in this case.

It remains to analyze the case when $\ell_1, \ell_4 = 0$, but $\ell_2, \ell_3 \neq 0$. As we have shown above, the summands with $(c_1, p) = 1$ do not contribute to the value of the sum except when the highest weight inequalities are satisfied. However, this is not sufficient because in the Iwasawa decomposition that gives rise to the sum, summands with $(c_1, p) = p$ do occur when $\ell_1 = 0$. Accordingly, we now examine the sum over such c_1 more closely. (Note that if $\ell_1 = 0$ then the n -th power residue symbol in c_1 is identically 1.)

Let $\text{ord}_p(c_1) = j$ with $j \geq 1$ and write $c_1 = p^j c'_1$. Substituting into the sum, the contribution becomes

$$\sum \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \psi \left(-p^{m_1 + j} \left(\frac{b_2 c'_1}{p^{\ell_3}} \right) + p^{m_3 + j} \left(\frac{c'_1 b_3}{p^{\ell_2}} \right) \right),$$

where the sum is over $c_i \pmod{p^{\ell_i}}$, $(c_i, p) = 1$, $i = 2, 3$, and over c'_1 modulo $p^{\ell_2 + \ell_3 - j}$ with $(c'_1, p) = 1$. We change $c_2 \mapsto c_2 c'_1$ and $c_3 \mapsto c_3 c'_1$; this removes c'_1 from the argument of ψ and introduces the power residue symbol $(c'_1 / p^{\ell_2 + \ell_3})$. The sum over c_2 , resp. c_3 , contributes

$$\begin{cases} h(\ell_2) & \text{if } m_1 + j \geq \ell_3, \\ \bar{g}(\ell_2) & \text{if } m_1 + j = \ell_3 - 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{resp.} \quad \begin{cases} h(\ell_3) & \text{if } m_3 + j \geq \ell_2, \\ \bar{g}(\ell_3) & \text{if } m_3 + j = \ell_2 - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here \bar{g} is the Gauss sum made from the conjugate n -th power residue symbol. Note that if n divides both ℓ_2 and ℓ_3 , these values are:

$$\begin{cases} \phi(p^{\ell_2}) & \text{if } m_1 + j \geq \ell_3, \\ -q^{\ell_2 - 1} & \text{if } m_1 + j = \ell_3 - 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{resp.} \quad \begin{cases} \phi(p^{\ell_3}) & \text{if } m_3 + j \geq \ell_2, \\ -q^{\ell_3 - 1} & \text{if } m_3 + j = \ell_2 - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\ell_3 - m_1 < \ell_2 - m_3$. The highest weight inequalities are satisfied unless $\ell_2 - m_3 > 1$. In that case, the sum over j , $1 \leq j \leq \ell_2 + \ell_3$, gives

$$h(\ell_2) \left(\bar{g}(\ell_3) h(\ell_2 + \ell_3) q^{-\ell_2 + m_3 + 1} + h(\ell_3) \left\{ 1 + \sum_{j=\ell_2 - m_3}^{\ell_2 + \ell_3 - 1} h(\ell_2 + \ell_3) q^{-j} \right\} \right). \quad (72)$$

Because of the presence of the functions h , the expression (72) is zero unless n divides both ℓ_2 and ℓ_3 . In that case, (72) is given by

$$\phi(p^{\ell_2}) \left((-q^{\ell_3 - 1}) \phi(p^{\ell_3 + m_3 + 1}) + \phi(p^{\ell_3}) \sum_{j=\ell_2 - m_3}^{\ell_2 + \ell_3} \phi(p^{\ell_2 + \ell_3 - j}) \right) = 0.$$

Similarly the sum is zero if $\ell_3 - m_1 > \ell_2 - m_3$.

This proves the description of the support of $S_{\ell, \mathbf{m}}$ given in Theorem 10.1. In the one case where the highest weight inequalities are not satisfied, that is, in the case $\ell_3 - m_1 = \ell_2 - m_3 > 1$, we obtain the contribution

$$\bar{g}(\ell_2) \bar{g}(\ell_3) h(\ell_2 + \ell_3) q^{-\ell_2 + m_3 + 1} + h(\ell_2) h(\ell_3) \left\{ 1 + \sum_{j=\ell_2 - m_3}^{\ell_2 + \ell_3 - 1} h(\ell_2 + \ell_3) q^{-j} \right\}.$$

This is zero unless:

- (1) n does not divide ℓ_2 and ℓ_3 but n divides $\ell_2 + \ell_3$; or
- (2) n divides ℓ_2 and ℓ_3 .

In case (1), the sums $\bar{g}(\ell_2)$ and $\bar{g}(\ell_3)$ are (up to a power of q) nontrivial conjugate Gauss sums; thus in this case the contribution is $q^{m_3 + \ell_3} \phi(p^{\ell_2 + \ell_3})$. In case (2), the contribution is

$$q^{\ell_2 + \ell_3 - 2} \phi(p^{\ell_3 + m_3 + 1}) + \phi(p^{\ell_2}) \phi(p^{\ell_3}) \sum_{j=\ell_2 - m_3}^{\ell_2 + \ell_3} \phi(p^{\ell_2 + \ell_3 - j}).$$

Since the sum telescopes, this simplifies to

$$q^{\ell_2 + \ell_3 - 2} \phi(p^{\ell_3 + m_3 + 1}) + q^{\ell_3 + m_3} \phi(p^{\ell_2}) \phi(p^{\ell_3}) = \phi(p^{\ell_2 + \ell_3 + m_3 + \ell_3}). \quad (73)$$

Note that this is equal to the contribution in case (1). Since $\ell_1 = \ell_4 = 0$, this expression agrees with (68). This completes the proof of Theorem 10.1.

Combining the result of (73) together with Lemma 10.4, we also obtain:

Proposition 10.5. *In case (b) of Theorem 10.1, with notation as in (65),*

$$S_{\ell, \mathbf{m}} = q^{\ell_2 + \ell_3 + 2\ell_4 - k - \ell_1} g_0(m_2, \ell_4) h(2\ell_1 + \ell_2 + \ell_3) \quad (74)$$

with $k = \ell_2 - m_3 = \ell_3 - m_1$ (and $g_0(m_2, 0) = 1$ by definition).

Proof. Combining the results of Lemma 10.4 with (73) in case (b) of Theorem 10.1 gives

$$S_{\ell, \mathbf{m}} = q^{\ell_2 + \ell_3 + 2\ell_4 - k} (h(\ell_2 + \ell_3) g_0(m_2, \ell_4) h(\ell_1) + \delta_n(\ell_1) p^{-\ell_1} h(2\ell_1 + \ell_2 + \ell_3) g_0(m_2, \ell_4)),$$

where $\delta_n(\ell_1) = 1$ if $\ell_1 > 0$ and n does not divide ℓ_1 , $\delta_n(\ell_1) = q^{-1}$ if $\ell_1 > 0$ and $n \mid \ell_1$, and $\delta_n(0) = 0$. This simplifies to the value in (74). \square

10.3. A vanishing theorem for Lusztig data outside a highest weight module.

The evaluation of the exponential sums $S_{\ell, \mathbf{m}}$ for $\ell = (\ell_1, \dots, \ell_4)$ with $\ell_i = \text{ord}_p(d_i)$ is finer than what is required to compute the Whittaker coefficient. Recall from (28) that the Whittaker coefficient is a Dirichlet series in $\widetilde{\mathfrak{D}}^{\rho_P}$, expressible as a product of the d_i according to Proposition 5.9. In the context of our \widetilde{GL}_4 example, this is given explicitly in (36) where $\mathfrak{D}^{\rho_P} = |d_1 d_2 d_3 d_4|^2$. Thus to assess the vanishing of the Dirichlet series at powers of a fixed prime p , we may sum over contributions of $H(d_1, d_2, d_3, d_4)$ with $d_i = p^{\ell_i}$ and such that $\ell_1 + \dots + \ell_4$ is a fixed constant. Indeed it suffices to analyze H since the remaining terms in the Whittaker coefficient in (28) depend only on \mathfrak{D} and not the individual d_j (since the S -Hilbert symbol $(p, p)_S = 1$). The following result demonstrates that while the prime power contributions to H , written as $S_{\ell, \mathbf{m}}$, may individually be non-zero for ℓ not belonging to the module of highest weight $\mathbf{m} + \rho$ for $GL_4(\mathbb{C})$, the total contribution of all such sums to the Whittaker coefficient is indeed zero.

Proposition 10.6. *Fix a positive integer k and non-negative 3-tuple $\mathbf{m} = (m_1, m_2, m_3)$. Suppose that $(\ell_1, \ell_2, \ell_3, \ell_4)$ fail to satisfy at least one of the inequalities (60)–(62). Then*

$$\sum_{\substack{(\ell_1, \ell_2, \ell_3, \ell_4) \\ \ell_1 + \ell_2 + \ell_3 + \ell_4 = k}} S_{\ell, \mathbf{m}} = 0.$$

Proof. By Theorem 10.1, the only vectors $(\ell_1, \ell_2, \ell_3, \ell_4)$ which fail one of (60)–(62) and have non-zero $S_{\ell, \mathbf{m}}$ are those with

$$0 \leq \ell_1 = \ell_4 \leq m_2 + 1, \quad \ell_2 - m_3 = \ell_3 - m_1 > 1,$$

so that (together with the condition $\ell_1 + \ell_2 + \ell_3 + \ell_4 = k$) ℓ_1, ℓ_2 , and ℓ_3 are uniquely determined by a choice of $\ell_4 \in [0, m_2 + 1]$.

For each choice of ℓ_4 , the summands may be evaluated using equation (74). They are all zero unless $n|k$. In the nonzero case, they are evaluated as follows. The sum over the range $0 \leq \ell_4 < m_2 + 1$ gives

$$\sum_{0 \leq \ell_4 < m_2 + 1} \phi(p^{\ell_4}) \phi(p^{3\ell_1 + 2\ell_3 + \ell_2 + m_3}) = \phi(p^{3\ell_1 + 2\ell_3 + \ell_2 + m_2 + m_3}).$$

If $\ell_4 = m_2 + 1$, then, using (74), the summand becomes $-\phi(p^{3\ell_1 + 2\ell_3 + \ell_2 + m_2 + m_3})$. The sum of these terms is zero, as claimed. \square

11. EVALUATING $S_{\ell, \mathbf{m}}$ FOR ℓ IN $\mathcal{B}_{\mathbf{m}+\rho}^\vee$

We continue in the context of the \widetilde{GL}_4 example, with fixed non-negative integer 3-tuple $\mathbf{m} = (m_1, m_2, m_3)$ and $S_{\ell, \mathbf{m}}$ as in (38). The Lusztig data ℓ for canonical basis elements in $\mathcal{B}_{\mathbf{m}+\rho}^\vee$ is thus defined as the set of non-negative integer 4-tuples satisfying (60)–(62). In this section, we compute the evaluation of $S_{\ell, \mathbf{m}}$ for all ℓ in $\mathcal{B}_{\mathbf{m}+\rho}^\vee$ according to cases.

11.1. Evaluation for positive Lusztig data. Suppose that all $\ell_i > 0$ and the highest weight inequalities (60), (61), (62) hold. There are four cases.

Case 1: $\ell_1 + \ell_3 \leq m_1 + \ell_4$, $\ell_1 + \ell_2 \leq m_3 + \ell_4$.

In this case, the ψ term is independent of c_1 and so the c_1 sum gives $p^{\ell_2+\ell_3}h(\ell_1)$. So the sum is:

$$q^{\ell_2+\ell_3+2\ell_4}h(\ell_1) \sum \left(\frac{c_2}{p^{\ell_2}}\right) \left(\frac{c_3}{p^{\ell_3}}\right) \left(\frac{c_4}{p^{\ell_4}}\right) \psi \left(-p^{m_1} \frac{b_4 c_3}{p^{\ell_3}} + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_2 b_4}{p^{\ell_2}}\right).$$

Changing $c_2 \mapsto c_2 c_4$ and $c_3 \mapsto c_3 c_4$ gives

$$q^{\ell_2+\ell_3+2\ell_4}h(\ell_1) \sum \left(\frac{c_2}{p^{\ell_2}}\right) \left(\frac{c_3}{p^{\ell_3}}\right) \left(\frac{c_4}{p^{\ell_2+\ell_3+\ell_4}}\right) \psi \left(-p^{m_1} \frac{c_3}{p^{\ell_3}} + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_2}{p^{\ell_2}}\right).$$

These sums are easily evaluated. The value in this case is:

$$S_{\ell, \mathbf{m}} = q^{2\ell_4} h(\ell_1) g(m_3, \ell_2) g(m_1, \ell_3) g(\ell_2 + \ell_3 + m_2, \ell_2 + \ell_3 + \ell_4). \quad (75)$$

Case 2: $\ell_1 + \ell_3 \leq m_1 + \ell_4$, $\ell_1 + \ell_2 > m_3 + \ell_4$.

By (61), this can only happen when $\ell_4 \leq \ell_1$ and $\ell_1 + \ell_2 = m_3 + \ell_4 + 1$. Also, since $\ell_1 + \ell_3 \leq m_1 + \ell_4$ and $\ell_4 \leq \ell_1$, it follows that $\ell_3 \leq m_1$. For given c_2, c_3, c_4 , we begin with the c_1 sum. This is

$$\sum_{c_1 \bmod \times p^{\ell_1+\ell_2+\ell_3}} \left(\frac{c_1}{p^{\ell_1}}\right) \psi \left(\frac{b_3 c_1}{p}\right).$$

Changing $c_1 \mapsto c_1 c_3$, this is easily evaluated. We find that

$$S_{\ell, \mathbf{m}} = q^{\ell_2+\ell_3+2\ell_4} g(\ell_1) \sum_{c_2, c_3, c_4} \left(\frac{c_2}{p^{\ell_2}}\right) \left(\frac{c_3}{p^{\ell_1+\ell_3}}\right) \left(\frac{c_4}{p^{\ell_4}}\right) \psi \left(p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_2 b_4}{p^{\ell_2}}\right).$$

Now change $c_2 \mapsto c_2 c_4$. The sum factors and each summand is easily evaluated. The value in this case is:

$$S_{\ell, \mathbf{m}} = q^{\ell_3+2\ell_4-\ell_1} g(\ell_1) g(m_3, \ell_2) h(\ell_1 + \ell_3) g(m_2 + \ell_2, \ell_2 + \ell_4). \quad (76)$$

Case 3: $\ell_1 + \ell_3 > m_1 + \ell_4$, $\ell_1 + \ell_2 \leq m_3 + \ell_4$.

This case is symmetric to Case 2. In view of (62), it occurs only when $\ell_4 \leq \ell_1$ and $\ell_1 + \ell_3 = m_1 + \ell_4 + 1$. Arguing as in Case 2, one sees that

$$S_{\ell, \mathbf{m}} = q^{\ell_2+2\ell_4-\ell_1} g(\ell_1) h(\ell_1 + \ell_2) g(m_1, \ell_3) g(m_2 + \ell_3, \ell_3 + \ell_4).$$

Case 4: $\ell_1 + \ell_3 > m_1 + \ell_4$, $\ell_1 + \ell_2 > m_3 + \ell_4$.

As before, the highest weight inequalities imply that $\ell_4 \leq \ell_1$ and that we have:

$$\ell_1 + \ell_2 = m_3 + \ell_4 + 1, \quad \ell_1 + \ell_3 = m_1 + \ell_4 + 1.$$

Note that this implies that $m_1 + \ell_2 = m_3 + \ell_3$ and also that $\ell_2 \leq m_3 + 1$, $\ell_3 \leq m_1 + 1$. We separate out two cases:

Case 4, Subcase A: In addition to the assumptions of Case 4, assume $\ell_4 < \ell_1$.

This additional condition implies that $\ell_2 \leq m_3$ and $\ell_3 \leq m_1$. Thus the sum reduces to

$$S_{\ell, \mathbf{m}} = q^{2\ell_4} \sum_{c_1, c_2, c_3, c_4} \left(\frac{c_1}{p^{\ell_1}}\right) \left(\frac{c_2}{p^{\ell_2}}\right) \left(\frac{c_3}{p^{\ell_3}}\right) \left(\frac{c_4}{p^{\ell_4}}\right) \psi \left(-\frac{b_2 c_1}{p} + p^{m_2} \frac{c_4}{p^{\ell_4}} + \frac{c_1 b_3}{p}\right).$$

Changing $c_2 \mapsto c_2 c_1$ and $c_3 \mapsto c_3 c_1$, the sum is easily evaluated, and gives

$$q^{2\ell_4} g(m_2, \ell_4) h(\ell_1 + \ell_2 + \ell_3) \bar{g}(\ell_2) \bar{g}(\ell_3). \quad (77)$$

The conjugate Gauss sums could be removed by exploiting additional identities since the contribution is 0 unless n divides $\ell_1 + \ell_2 + \ell_3$.

Case 4, Subcase B: In addition to Case 4, we have $\ell_4 = \ell_1$.

Now we have

$$S_{\ell, \mathbf{m}} = q^{2\ell_4} \sum_{c_1, c_2, c_3, c_4} \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(-\frac{b_2 c_1 + b_4 c_3}{p} + p^{m_2} \frac{c_4}{p^{\ell_4}} + \frac{c_1 b_3 + c_2 b_4}{p} \right). \quad (78)$$

The subsum with $c_2 \equiv c_3 \pmod{p}$ is easily evaluated as the ψ function reduces to $\psi(c_4 p^{m_2 - \ell_4})$ and $(c_3/p^{\ell_3}) = (c_2/p^{\ell_3})$. This subsum gives

$$q^{\ell_2 + \ell_3 + 2\ell_4 - 1} g(m_4, \ell_4) h(\ell_2 + \ell_3) h(\ell_1). \quad (79)$$

The remaining term is the subsum with $c_3 = c_2 - a$ where $(a, p) = 1$. (Also $a \not\equiv c_2 \pmod{p}$ as $(p, c_3) = 1$.) Since in that case,

$$b_3 - b_2 \equiv c_3^{-1} - c_2^{-1} = (c_2 - c_3) c_2^{-1} c_3^{-1} \equiv a b_2 b_3 \pmod{p},$$

this subsum reduces to:

$$q^{2\ell_4} \sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(\frac{c_1 a b_2 b_3}{p} + p^{m_2} \frac{c_4}{p^{\ell_4}} + \frac{a b_4}{p} \right).$$

After changing $c_1 \mapsto c_1 c_2 c_3 a^{-1}$ (where a^{-1} is an inverse of a modulo p) and evaluating the c_1 sum, we arrive at

$$q^{\ell_2 + \ell_3 + 2\ell_4} g(\ell_1) \sum_{c_2, a, c_4} \left(\frac{c_2}{p^{\ell_1 + \ell_2}} \right) \left(\frac{c_2 - a}{p^{\ell_1 + \ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \left(\frac{a}{p^{\ell_1}} \right)^{-1} \psi \left(p^{m_2} \frac{c_4}{p^{\ell_4}} + \frac{a b_4}{p} \right).$$

Here we regard the sum as over c_2 , c_4 , and a . Changing $c_2 \mapsto a c_2$, we get a sum over c_2 , a , c_4 where now $c_2 \not\equiv 1 \pmod{p}$:

$$q^{\ell_2 + \ell_3 + 2\ell_4} g(\ell_1) \sum_{c_2, a, c_4} \left(\frac{c_2}{p^{\ell_1 + \ell_2}} \right) \left(\frac{c_2 - 1}{p^{\ell_1 + \ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \left(\frac{a}{p^{\ell_3}} \right) \psi \left(p^{m_2} \frac{c_4}{p^{\ell_4}} + \frac{a b_4}{p} \right).$$

Now we may change $a \mapsto a c_4$ and evaluate the a and c_4 sums. This gives

$$q^{\ell_2 + 2\ell_4} g(\ell_1) g(\ell_3 + m_2, \ell_3 + \ell_4) g(\ell_3) \sum_{\substack{c_2 \pmod{p^{\ell_2}} \\ c_2 \not\equiv 0, 1 \pmod{p}}} \left(\frac{c_2}{p^{\ell_1 + \ell_2}} \right) \left(\frac{c_2 - 1}{p^{\ell_1 + \ell_3}} \right).$$

The remaining sum in the expression above is a Jacobi sum. If $\ell_1 + \ell_2$, $\ell_1 + \ell_3$ are both zero modulo n then the sum takes value $(q-2)q^{\ell_2-1}$. If exactly one of $\ell_1 + \ell_2$, $\ell_1 + \ell_3$ is zero modulo n the sum is $-q^{\ell_2-1}$. If $\ell_1 + \ell_2$, $\ell_1 + \ell_3$ are both nonzero modulo n but $2\ell_1 + \ell_2 + \ell_3 \equiv 0 \pmod{n}$ then the sum is of the form $J(\chi, \chi^{-1})$ with χ not the trivial character, and gives $-q^{\ell_2-1}$. If $\ell_1 + \ell_2$, $\ell_1 + \ell_3$, $2\ell_1 + \ell_2 + \ell_3$ are all nonzero modulo n then the sum is a quotient of Gauss sums (times a power of q):

$$q^{\ell_2} \frac{g(\ell_1 + \ell_2) g(\ell_1 + \ell_3)}{g(2\ell_1 + \ell_2 + \ell_3)}.$$

Combining this with the other terms, we see that the full sum for Case 4, Subcase B is equal to:

$$q^{\ell_2+\ell_3+2\ell_4-1} g(m_4, \ell_4) h(\ell_2 + \ell_3) h(\ell_1) + q^{\ell_2+2\ell_4} g(\ell_1) g(\ell_3 + m_2, \ell_3 + \ell_4) g(\ell_3) \times \begin{cases} (q-2)q^{\ell_2-1} & \text{if } \ell_1 + \ell_2, \ell_1 + \ell_3 \equiv 0 \pmod n \\ -q^{\ell_2-1} & \text{if exactly one of } \ell_1 + \ell_2, \ell_1 + \ell_3 \text{ is } \equiv 0 \pmod n \\ -q^{\ell_2-1} & \text{if } \ell_1 + \ell_2, \ell_1 + \ell_3 \not\equiv 0 \pmod n, \text{ and} \\ & 2\ell_1 + \ell_2 + \ell_3 \equiv 0 \pmod n \\ q^{\ell_2} \frac{g(\ell_1+\ell_2) g(\ell_1+\ell_3)}{g(2\ell_1+\ell_2+\ell_3)} & \text{if } \ell_1 + \ell_2, \ell_1 + \ell_3, 2\ell_1 + \ell_2 + \ell_3 \not\equiv 0 \pmod n. \end{cases} \quad (80)$$

This last case demonstrates the complexity of $S_{\ell, \mathbf{m}}$ for Lusztig data ℓ lying simultaneously on the maximal hyperplanes $\ell_2 = m_3 + 1$ and $\ell_1 = m_1 + 1$.

11.2. Evaluation with at least one ℓ_i equal to 0. We handle the remaining cases when the highest weight inequalities (60), (61), (62) are satisfied.

First, suppose that $\ell_1 = 0$, $\ell_2 \leq m_3 + 1$, $\ell_3 \leq m_1 + 1$.

If, in addition, $\ell_4 > 0$, then the sum is evaluated as follows. Note that since $\ell_1 = 0$, the c_1 sum is modulo $p^{\ell_2+\ell_3}$ without a relative primality condition, and there is no residue symbol in c_1 . The function ψ is independent of c_1 so the c_1 sum gives $p^{\ell_2+\ell_3}$. After changing $c_i \mapsto c_i c_4$, $i = 2, 3$, the evaluation proceeds similarly to Case 2. It gives:

$$S_{\ell, \mathbf{m}} = q^{2\ell_4} g(m_3, \ell_2) g(m_1, \ell_3) g(m_2 + \ell_2 + \ell_3, \ell_2 + \ell_3 + \ell_4). \quad (81)$$

(Here $g(m, 0) = 1$ in case ℓ_2 or ℓ_3 is zero.)

Suppose instead that $\ell_1 = \ell_4 = 0$. If $\ell_2 = \ell_3 = 0$, then the sum is of course 1. Otherwise, we break the sum into two pieces. First, if $c_1 \equiv 0 \pmod p$ then the character $\psi(\cdot)$ is identically 1, so this piece gives

$$q^{\ell_2+\ell_3-1} h(\ell_2) h(\ell_3).$$

Second, there is the sum over c_1 such that $(c_1, p) = 1$. Changing $c_2 \mapsto c_1 c_2$, $c_3 \mapsto c_1 c_3$ we obtain

$$\begin{aligned} & \sum \left(\frac{c_1}{p^{\ell_2+\ell_3}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \psi \left(-p^{m_1} \frac{b_2}{p^{\ell_3}} + p^{m_3} \frac{b_3}{p^{\ell_2}} \right) \\ &= h(\ell_2 + \ell_3) \begin{cases} \bar{g}(m_1 + \ell_2 - \ell_3, \ell_2) \bar{g}(m_3 + \ell_3 - \ell_2, \ell_3) & \text{if } \ell_2, \ell_3 > 0 \\ h(\ell_3) & \text{if } \ell_2 = 0, \ell_3 > 0, \\ h(\ell_2) & \text{if } \ell_2 > 0, \ell_3 = 0. \end{cases} \quad (82) \end{aligned}$$

(Note that by the highest weight inequalities,

$$\ell_2 \leq m_3 + 1 \quad \ell_3 \leq m_1 + 1,$$

so that the first entries of the Gauss sums above are non-negative.)

Next, suppose that $\ell_1 > 0$ but $\ell_2 = 0$. The sum becomes

$$q^{2\ell_4} \sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(-p^{m_1} \frac{b_4 c_3}{p^{\ell_3}} + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_1 b_3 p^{\ell_4}}{p^{\ell_1}} \right).$$

If $\ell_3, \ell_4 > 0$, then changing $c_1 \mapsto c_1 c_3$ and then $c_3 \mapsto c_3 c_4$ gives

$$q^{2\ell_4} \sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_3}{p^{\ell_1+\ell_3}} \right) \left(\frac{c_4}{p^{\ell_1+\ell_3+\ell_4}} \right) \psi \left(-p^{m_1} \frac{c_3}{p^{\ell_3}} + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_1 p^{\ell_4}}{p^{\ell_1}} \right) \\ = q^{2\ell_4-2\ell_1} g(m_3 + \ell_4, \ell_1) g(\ell_1 + m_1, \ell_1 + \ell_3) g(\ell_1 + \ell_3 + m_2, \ell_1 + \ell_3 + \ell_4). \quad (83)$$

In the remaining cases, a similar argument gives

$$\begin{cases} q^{-\ell_1+\ell_3} g(m_3, \ell_1) h(\ell_1 + \ell_3) & \text{if } \ell_3 > 0, \ell_4 = 0, \\ q^{2\ell_4} g(m_2, \ell_4) h(\ell_1) & \text{if } \ell_3 = 0, \ell_4 > 0, \\ h(\ell_1) & \text{if } \ell_3 = \ell_4 = 0. \end{cases} \quad (84)$$

For a third possibility, suppose that $\ell_1, \ell_2 > 0$ but $\ell_3 = 0$. The sum is

$$p^{2\ell_4} \sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_4}{p^{\ell_4}} \right) \psi \left(-p^{m_1} \frac{b_2 c_1 p^{\ell_4}}{p^{\ell_1}} + p^{m_2} \frac{c_4}{p^{\ell_4}} + p^{m_3} \frac{c_2 b_4}{p^{\ell_2}} \right).$$

Proceeding as above, this gives

$$\begin{cases} q^{2\ell_4-2\ell_1} g(m_1 + \ell_4, \ell_1) g(\ell_1 + m_3, \ell_1 + \ell_2) g(\ell_1 + \ell_2 + m_2, \ell_1 + \ell_2 + \ell_4) & \text{if } \ell_4 > 0 \\ q^{-\ell_1+\ell_2} g(m_1, \ell_1) h(\ell_1 + \ell_2) & \text{if } \ell_4 = 0 \end{cases}$$

Last suppose that $\ell_1, \ell_2, \ell_3 > 0$ but $\ell_4 = 0$. The sum becomes

$$\sum \left(\frac{c_1}{p^{\ell_1}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \psi \left(-p^{m_1} \frac{b_2 c_1}{p^{\ell_1+\ell_3}} + p^{m_3} \frac{c_1 b_3}{p^{\ell_1+\ell_2}} \right).$$

Changing $c_2 \mapsto c_2 c_1$ and $c_3 \mapsto c_3 c_1$, we obtain

$$\sum \left(\frac{c_1}{p^{\ell_1+\ell_2+\ell_3}} \right) \left(\frac{c_2}{p^{\ell_2}} \right) \left(\frac{c_3}{p^{\ell_3}} \right) \psi \left(-p^{m_1} \frac{b_2}{p^{\ell_1+\ell_3}} + p^{m_3} \frac{b_3}{p^{\ell_1+\ell_2}} \right).$$

This gives

$$h(\ell_1 + \ell_2 + \ell_3) \bar{g}(m_1 + \ell_2 - \ell_1 - \ell_3, \ell_2) \bar{g}(m_3 + \ell_3 - \ell_1 - \ell_2, \ell_3). \quad (85)$$

(Note that by the highest weight inequalities,

$$\ell_1 + \ell_2 \leq m_3 + 1 \quad \ell_1 + \ell_3 \leq m_1 + 1,$$

so that the first entries of the Gauss sums above are non-negative.)

11.3. BZL patterns and the exponential sums. In Section 8.3, we presented Berenstein-Zelevinsky and Littelmann's results on string data: for any highest weight representation of G^\vee , the string data for the corresponding canonical basis vectors are integer lattice points in a polytope. In this section, we show that for our \widetilde{GL}_4 example, the evaluation of the exponential sum $S_{\ell, \mathbf{m}}$ with Lusztig data $\ell = (\ell_1, \dots, \ell_4)$ can be described using the corresponding *string* data according to (59). To facilitate this description, we place the string data corresponding to ℓ in a simple one-row array:

$$\boxed{\max(\ell_1, \ell_4) + \ell_2 + \ell_3 \mid \ell_1 + \ell_3 \mid \ell_1 + \ell_2 \mid \min(\ell_1, \ell_4)}. \quad (86)$$

We refer to the resulting array as a “BZL pattern” and use the notation \mathcal{P}_ℓ . Arrays with multiple rows appear in [27] corresponding to a sequence of relatively maximal parabolics from G^\vee down to the Borel subgroup. These played a prominent role in our earlier investigations of metaplectic Eisenstein series in type A with Daniel Bump ([10, 11]) and in proofs

and conjectures of other types (e.g., [3, 17, 20, 21]). A product of Gauss sums is attached to any such BZL pattern as follows.

Each integer entry c of a BZL pattern is constrained by inequalities depending on the other entries and the highest weight encoded by \mathbf{m} . For example, the entry $c = \min(\ell_1, \ell_4)$ above is constrained by $0 \leq c \leq m_2 + 1$ according to (53) and (54). We refer to the element c as being “maximal” if the upper inequality is an equality and “minimal” if the lower inequality is an equality. Then define the function

$$\gamma(a) = \begin{cases} g(a), & \text{if } a \text{ is maximal,} \\ h(a), & \text{if } a \text{ is neither maximal nor minimal,} \\ q^a, & \text{if } a \text{ is minimal,} \\ 0, & \text{if } a \text{ is both maximal and minimal.} \end{cases}$$

Given a BZL pattern \mathcal{P}_ℓ , we then define the *standard contribution*, $G(\mathcal{P}_\ell)$, as follows:

$$G(\mathcal{P}_\ell) = \prod_{a \in \mathcal{P}_\ell} \gamma(a).$$

In [10], the standard contribution exactly matched the exponential sum for covers of $G = GL_{r+1}$ and maximal parabolic omitting the root α_1 . The following result shows that the standard contribution for \mathcal{P}_ℓ “generically” agrees with the evaluation of the prime-powered exponential sum $S_{\ell, \mathbf{m}}$, though comparison with the results of Sections 11.1 and 11.2 show it fails to agree in general.

Theorem 11.1. *Let \mathcal{P}_ℓ be the BZL pattern corresponding to ℓ as in (86). Then*

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = G(\mathcal{P}_\ell),$$

if $\ell = (\ell_1, \dots, \ell_4)$ avoids the following cases:

- $\ell_1 = \ell_4 = m_2 + 1$,
- $\ell_1 = m_2 + 1$ and $\ell_4 = 0$,
- $\ell_2 = m_3 + 1, \ell_3 = m_1 + 1$, and $\ell_1 = \ell_4$,
- $\ell_1 = \ell_4 = 0$, and
- $\ell_2 = \ell_3 = 0$.

In fact, the theorem holds for some Lusztig data ℓ in the excluded cases, but because it is difficult to state succinctly and in some cases depends on the divisibility of the ℓ_i by n , we omit the precise description of where the standard contribution fails. This information may be extracted from the proof.

Proof. First, we handle the case of positive Lusztig data. So suppose that all $\ell_i > 0$. Our numbering of cases corresponds to that of Section 11.1.

Case 1: $\ell_1 + \ell_3 \leq m_1 + \ell_4$, $\ell_1 + \ell_2 \leq m_3 + \ell_4$.

According to (75),

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = h(\ell_1) g(\ell_1 + m_3, \ell_1 + \ell_2) g(\ell_1 + m_1, \ell_1 + \ell_3) g(\ell_2 + \ell_3 + m_2, \ell_2 + \ell_3 + \ell_4). \quad (87)$$

If $\ell_1 \leq \ell_4$ then the original four highest weight inequalities reduce to

$$\ell_1 \leq m_2 + 1 \quad \ell_2 \leq m_3 + 1 \quad \ell_3 \leq m_1 + 1 \quad \ell_4 \leq m_2 + 1.$$

If $\ell_1 < \ell_4$ then the leftmost inequality is never an equality, and so the evaluation (87) matches the value of the standard contribution $G(\mathcal{P}_\ell)$ for the BZL pattern \mathcal{P}_ℓ :

$$\boxed{\ell_2 + \ell_3 + \ell_4} \boxed{\ell_1 + \ell_3} \boxed{\ell_1 + \ell_2} \boxed{\ell_1} \quad (88)$$

in all cases.

If $\ell_1 > \ell_4$ then the highest weight inequalities imply that $\ell_2 \leq m_3$, $\ell_3 \leq m_1$. Thus

$$\begin{aligned} q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} &= h(\ell_1) h(\ell_1 + \ell_2) h(\ell_1 + \ell_3) g(\ell_2 + \ell_3 + m_2, \ell_2 + \ell_3 + \ell_4) \\ &= h(\ell_1 + \ell_2 + \ell_3) h(\ell_1 + \ell_2) h(\ell_1 + \ell_3) g(m_2, \ell_4). \end{aligned} \quad (89)$$

(The ℓ_2 and ℓ_3 may be moved here since the value is identically zero unless n divides ℓ_2 and ℓ_3 .) This again matches the value $G(\mathcal{P}_\ell)$ for the BZL pattern \mathcal{P}_ℓ of form

$$\boxed{\ell_1 + \ell_2 + \ell_3} \boxed{\ell_1 + \ell_3} \boxed{\ell_1 + \ell_2} \boxed{\ell_4}. \quad (90)$$

Note that if $\ell_1 = \ell_4 = m_2 + 1$ then the outer terms in the BZL pattern contribute $g(\ell_1 + \ell_2 + \ell_3)g(\ell_4)$. This does not appear in the evaluation in (87). However, the terms $\ell_1 = \ell_4 < m_2 + 1$ indeed match $G(\mathcal{P}_\ell)$.

Case 2: $\ell_1 + \ell_3 \leq m_1 + \ell_4$, $\ell_1 + \ell_2 > m_3 + \ell_4$.

Recall that by (61), this can only happen when $\ell_4 \leq \ell_1$ and $\ell_1 + \ell_2 = m_3 + \ell_4 + 1$, so the BZL pattern is (90) and the entry $\ell_1 + \ell_2$ is maximal, while $\ell_1 + \ell_3$ is not maximal.

Assume further that in fact $\ell_4 < \ell_1$. Then since $\ell_1 + \ell_2 = m_3 + \ell_4 + 1$, we see that $\ell_2 \leq m_3$. Thus according to (76),

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = q^{\ell_1 + \ell_3} g(\ell_1) h(\ell_2) h(\ell_1 + \ell_3) g(m_2 + \ell_2, \ell_2 + \ell_4) \quad (91)$$

$$= g(\ell_1 + \ell_2) h(\ell_1 + \ell_2 + \ell_3) h(\ell_1 + \ell_3) g(m_2, \ell_4). \quad (92)$$

Since $\ell_4 < \ell_1$ implies $\ell_1 + \ell_2 + \ell_3$ is not maximal in the pattern \mathcal{P}_ℓ , the value of $q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}}$ exactly matches the standard contribution of the BZL pattern.

Once again, if $\ell_1 = \ell_4 = m_2 + 1$, so that $\ell_1 + \ell_2 + \ell_3$ is maximal, we do not match

$$G(\mathcal{P}_\ell) = g(\ell_1 + \ell_2) g(\ell_1 + \ell_2 + \ell_3) h(\ell_1 + \ell_3) g(m_2, \ell_4).$$

Case 3 is symmetric with Case 2, and the discussion is similar. In particular, the BZL pattern gives the same value unless $\ell_1 = \ell_4 = m_2 + 1$.

Case 4: $\ell_1 + \ell_3 > m_1 + \ell_4$, $\ell_1 + \ell_2 > m_3 + \ell_4$

If $\ell_1 \neq \ell_4$, this is Subcase A and we must have $\ell_4 < \ell_1$. Thus the associated BZL pattern \mathcal{P}_ℓ is as in (90). In Subcase A, one has

$$\ell_1 + \ell_2 = m_3 + \ell_4 + 1, \quad \ell_1 + \ell_3 = m_1 + \ell_4 + 1,$$

so that $\ell_1 + \ell_2$ and $\ell_1 + \ell_3$ are maximal. Also, $\ell_4 \leq m_2 + 1$ implies that the highest weight inequality $\max(\ell_1, \ell_4) \leq m_2 + 1 + 2\ell_1 - 2\min(\ell_1, \ell_4)$ is always strict, so that $\ell_1 + \ell_2 + \ell_3$ is never maximal, and contributes $h(\ell_1 + \ell_2 + \ell_3)$ to $G(\mathcal{P}_\ell)$. By (77),

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = q^{2\ell_1} g(m_2, \ell_4) h(\ell_1 + \ell_2 + \ell_3) \bar{g}(\ell_2) \bar{g}(\ell_3).$$

However, if $n|(a+b)$ then $\bar{g}(a) = g(b)q^{a-b}$. Thus we find that

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = g(m_2, \ell_4) h(\ell_1 + \ell_2 + \ell_3) g(\ell_1 + \ell_3) g(\ell_1 + \ell_2),$$

which matches the standard contribution of the corresponding BZL pattern.

(In contrast to the previous three cases, the condition $\ell_1 = \ell_4 \neq m_2 + 1$ in Subcase 4B is not sufficient to guarantee that the evaluation of $S_{\ell, \mathbf{m}}$ matches the standard contribution of the BZL pattern. Subcase 4B requires $\ell_2 = m_3 + 1$ and $\ell_3 = m_1 + 1$, and we must avoid a larger subset of these cases with $\ell_1 = \ell_4$.)

Remaining Cases: At least one of the $\ell_i = 0$.

If $\ell_1 = 0$, $\ell_4 > 0$, then the corresponding BZL pattern is

$$\boxed{\ell_2 + \ell_3 + \ell_4} \boxed{\ell_3} \boxed{\ell_2} \boxed{0}$$

Thus the standard contribution indeed matches (81), which gives

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = g(m_3, \ell_2) g(m_1, \ell_3) g(m_2 + \ell_2 + \ell_3, \ell_2 + \ell_3 + \ell_4).$$

Next, suppose that $\ell_1 > 0$ but $\ell_2 = 0$. Then combining (83) and (84),

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = \begin{cases} g(m_3 + \ell_4, \ell_1) g(\ell_1 + m_1, \ell_1 + \ell_3) \\ \quad \times g(\ell_1 + \ell_3 + m_2, \ell_1 + \ell_3 + \ell_4) & \text{if } \ell_3, \ell_4 > 0, \\ q^{\ell_1 + \ell_3} g(m_3, \ell_1) h(\ell_1 + \ell_3) & \text{if } \ell_3 > 0, \ell_4 = 0, \\ q^{2\ell_1} g(m_2, \ell_4) h(\ell_1) & \text{if } \ell_3 = 0, \ell_4 > 0, \\ q^{2\ell_1} h(\ell_1) & \text{if } \ell_3 = \ell_4 = 0. \end{cases}$$

It can be verified on a case-by-case analysis that this matches the standard contribution of the BZL pattern unless $\ell_2 = \ell_3 = 0$ and either $\ell_1 > \ell_4 > 0$ or $\ell_4 = m_2 + 1 > \ell_1$. The case when $\ell_1, \ell_2 > 0$ but $\ell_3 = 0$ may be handled symmetrically to the top two cases above.

Finally, suppose $\ell_1, \ell_2, \ell_3 > 0$ and $\ell_4 = 0$, then from (85),

$$q^{2\ell_1 - 2\ell_4} S_{\ell, \mathbf{m}} = q^{2\ell_1} h(\ell_1 + \ell_2 + \ell_3) \bar{g}(m_1 + \ell_2 - \ell_1 - \ell_3, \ell_2) \bar{g}(m_3 + \ell_3 - \ell_1 - \ell_2, \ell_3).$$

This may be analyzed depending on whether $\ell_1 + \ell_2$ and $\ell_1 + \ell_3$ appearing in (90) are maximal. If either are maximal, the corresponding Gauss sum is of form $\bar{g}(a)$ for some integer a . As this contribution is zero unless n divides $\ell_1 + \ell_2 + \ell_3$, the relation $\bar{g}(a) = g(b)q^{a-b}$ for n dividing $a + b$ may be used to convert the above conjugate Gauss sums into non-conjugate Gauss sums. If $\ell_1 + \ell_3$ or $\ell_1 + \ell_2$ is not maximal, then the associated conjugate Gauss sums are of the form $h(a)$ for some integer a . In either case, the evaluation above matches the standard contribution of the BZL pattern \mathcal{P} *unless* the entry $\ell_1 + \ell_2 + \ell_3$ is maximal (i.e. $\ell_1 = m_2 + 1$) in which case $G(\mathcal{P})$ contains the Gauss sum $g(\ell_1 + \ell_2 + \ell_3)$ not $h(\ell_1 + \ell_2 + \ell_3)$. \square

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